

DEFORMATION OF SCALAR CURVATURE AND VOLUME

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ABSTRACT. The stationary points of the total scalar curvature functional on the space of unit volume metrics on a given closed manifold are known to be precisely the Einstein metrics. One may consider the modified problem of finding stationary points for the volume functional on the space of metrics whose scalar curvature is equal to a given constant. In this paper, we localize a condition satisfied by such stationary points to smooth bounded domains. The condition involves a generalization of the *static* equations, and we interpret solutions (and their boundary values) of this equation variationally. On domains carrying a metric that does not satisfy the condition, we establish a local deformation theorem that allows one to achieve simultaneously small prescribed changes of the scalar curvature and of the volume by a compactly supported variation of the metric. We apply this result to obtain a localized gluing theorem for constant scalar curvature metrics in which the total volume is preserved. Finally, we note that starting from a counterexample of Min-Oo's conjecture such as that of Brendle-Marques-Neves, counterexamples of arbitrarily large volume and different topological types can be constructed.

1. INTRODUCTION

Let M be a closed manifold with dimension at least three, \mathcal{M} the cone of Riemannian metrics on M , and $\mathcal{M}^c \subset \mathcal{M}$ the subset of Riemannian metrics with constant scalar curvature c . Let $V(g) = \text{vol}(M, g)$ be the volume of a metric $g \in \mathcal{M}$, and let $R(g)$ be its scalar curvature. For $c \neq 0$, critical points of the restricted volume map $V_c : \mathcal{M}^c \rightarrow (0, \infty)$ are precisely stationary points of the total scalar curvature $\mathcal{R}(g) = \int_M R(g) d\mu_g$ restricted to \mathcal{M}^c . (Note that the total scalar curvature is a topological invariant in dimension two.) Critical metrics for V_c are special, as they admit non-trivial solutions (f, κ) to the overdetermined-elliptic system $L_g^* f = \kappa g$. Here, L_g is the linearization of the scalar curvature operator, L_g^* is its formal adjoint, and κ is a constant. We make this precise in Theorem 2.3.

In this paper, we localize the above analysis to the case where the metric deformations are supported on the closure of a bounded domain $\Omega \subset M$. In Theorem 1.1, we show that when the metric g does not admit non-trivial solutions to $L_g^* f = \kappa g$, then one can achieve simultaneously a prescribed perturbation of the scalar curvature that is compactly supported in Ω and a prescribed perturbation of the volume by a small deformation of the metric in $\overline{\Omega}$.

The obstruction to finding such a deformation of the metric is the existence of a non-trivial solution (f, κ) of the system $L_g^* f = \kappa g$ on Ω . If such a non-trivial

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solution (f, κ) exists, we call the metric g *V-static* with *V-static potential* f . This condition is a mild generalization of the *static* equation $L_g^* f = 0$, cf. [7]. A metric g is called *static* if the static equation admits a non-trivial solution f , in which case f is called a *static potential* for g . In Theorem 2.3, we provide a variational characterization of *V-static* metrics, emphasizing the role of the boundary values of a *V-static* potential. The case where $\kappa \neq 0$ and the *V-static* potential vanishes on the boundary was studied in [20], where an interesting volume comparison result (stated here as Theorem 2.4) was proved. We include a new proof of this result from [20] that actually leads to a slightly stronger result.

We now give a precise statement of the local deformation theorem. Let h be a symmetric $(0, 2)$ -tensor on M . The linearization L_g of the scalar curvature map $R : \mathcal{M} \rightarrow \mathbb{R}$ is $L_g(h) = -\Delta_g(\text{tr}_g h) + \text{div}_g \text{div}_g h - h \cdot \text{Ric}(g)$, and its formal L^2 -adjoint is $L_g^* f = -(\Delta_g f)g + \nabla_g^2 f - f \text{Ric}(g)$. (Our convention is that $\Delta_g f = \text{tr}_g(\nabla_g^2 f)$.) The variation of the volume map $V : \mathcal{M} \rightarrow (0, \infty)$ is $DV_g(h) = \frac{1}{2} \int_M \text{tr}_g(h) d\mu_g$. Let $\Theta(g) := (R(g), V(g))$. Let $\mathcal{S}_g(h) = D\Theta_g(h) = (L_g(h), DV_g(h))$. Its formal adjoint is then $\mathcal{S}_g^*(f, a) = L_g^* f + \frac{a}{2}g$. Thus *V-static* potentials correspond precisely to non-trivial elements in the kernel of \mathcal{S}_g^* .

The Banach spaces $\mathcal{B}_0 = \mathcal{B}_0(\Omega) \subset C^{0,\alpha}(\text{Sym}^2(T^*\Omega)) \times \mathbb{R}$ and $\mathcal{B}_2 = \mathcal{B}_2(\Omega) \subset C^{2,\alpha}(\text{Sym}^2(T^*\Omega))$ and their respective norms $\|\cdot\|_0$ and $\|\cdot\|_2$ that appear in the statement of the following theorem are introduced in Section 3.4.

Theorem 1.1. *Let $(\bar{\Omega}, g)$ be a compact $C^{4,\alpha}$ Riemannian manifold¹ of dimension $n \geq 2$ with boundary. Let Ω be the manifold interior of $\bar{\Omega}$. Assume that the equation $\mathcal{S}_g^*(f, a) = 0$ has no non-trivial solutions $(f, a) \in C^2(\Omega) \times \mathbb{R}$. There exist $\epsilon, C > 0$ so that for any $(\sigma, \tau) \in \mathcal{B}_0$ with $\|(\sigma, \tau)\|_0 < \epsilon$ there is a metric γ on $\bar{\Omega}$ so that $R(\gamma) = R(g) + \sigma$, $V(\gamma) = V(g) + \tau$. In fact, $\gamma - g \in \mathcal{B}_2$ and $\|\gamma - g\|_2 \leq C\|(\sigma, \tau)\|_0$. In particular, $\gamma - g$ can be extended by 0 as a $C^{2,\alpha}$ tensor across the boundary of $\bar{\Omega}$.*

The following version of Theorem 1.1 includes the dependence on the metric and higher order regularity:

Theorem 1.2. *Let $k \geq 4$. Let $(\bar{\Omega}, g_0)$ be a compact $C^{k,\alpha}$ Riemannian manifold of dimension $n \geq 2$ with boundary, and let Ω be the manifold interior of $\bar{\Omega}$. Assume that the equation $\mathcal{S}_{g_0}^*(f, a) = 0$ has no non-trivial solutions $(f, a) \in C^2(\Omega) \times \mathbb{R}$. Let $\Omega_0 \subset \Omega$ be a non-empty open set that is compactly contained in Ω . There exists an open neighborhood U of g_0 in $C^{k,\alpha}(\bar{\Omega})$ and $\epsilon, C > 0$ such that for any $g \in U$, $\tau \in \mathbb{R}$, and $\sigma \in C^{k-4,\alpha}(\bar{\Omega})$ with support in Ω_0 and with $|\tau| + \|\sigma\|_{C^{k-4,\alpha}} < \epsilon$, there is a $C^{k-2,\alpha}$ metric γ on $\bar{\Omega}$ so that $\text{supp}(\gamma - g)$ is compactly contained in Ω , such that $\|\gamma - g\|_{C^{k-2,\alpha}} \leq C(|\tau| + \|\sigma\|_{C^{k-4,\alpha}})$, and such that $R(\gamma) = R(g) + \sigma$, $V(\gamma) = V(g) + \tau$. If g and σ are smooth, we can arrange for γ to be smooth as well.*

Now we present several examples to illustrate the static and *V-static* conditions. Our convention is that λ is an eigenvalue for a Schrödinger operator $T = \Delta_g + w$ with non-zero eigenfunction u if $T(u) = -\lambda u$.

¹Given an integer $k \geq 1$ and $\alpha \in (0, 1)$, a $C^{k,\alpha}$ Riemannian manifold (M, g) consists of a smooth manifold M , possibly with non-empty boundary, and a tensor field $g \in C^{k,\alpha}(\text{Sym}^2(T^*M))$ that is everywhere positive definite.

Example 1.3. Recall that a metric g is critical for V_{-1} on a closed manifold M precisely when g is Einstein with $R(g) = -1$. We show that this is equivalent to being V -static with $R(g) = -1$. Indeed, in the Einstein case, $(f, \kappa) = (1, 1/n)$ satisfies the critical equation $L_g^* f = \kappa g$. Conversely, if $R(g) = -1$ and if (M, g) admits a non-trivial solution (f, κ) of $L_g^* f = \kappa g$, we first obtain that $-(n-1)\Delta_g f + f = n\kappa$ by taking the trace. Since $((n-1)\Delta_g - 1)$ is invertible, we conclude that $\kappa \neq 0$ and that $f = n\kappa$ is a solution. Plugging this back into the critical equation we obtain that $-n\kappa \text{Ric}(g) = \kappa g$. Thus g is Einstein.

The same argument shows that when $R(g) = 1$ and $\frac{1}{n-1}$ is not an eigenvalue of the Laplacian, then g is V -static if and only if g is Einstein.

Example 1.4. A scalar-flat V -static metric g on a closed manifold M is Ricci-flat. To see this, let (f, κ) be a non-trivial solution of $L_g^* f = \kappa g$. Taking the trace of this equation, we see that $\kappa = 0$ and that f is equal to a non-zero constant. Using this information in the equation $L_g^* f = \kappa g$, we obtain that (M, g) is Ricci-flat. Conversely, every Ricci-flat metric on M is V -static and the space of solutions (f, κ) of $L_g^* f = \kappa g$ is spanned by $(1, 0)$. Scaling changes the volume of a Ricci-flat metric at a nonzero rate while leaving the metric Ricci-flat. Therefore a Ricci-flat metric cannot be critical for V_0 .

Example 1.5. Consider the metric $g = (n-2)^{-1}g_{\mathbb{S}^1} + g_{\mathbb{S}^{n-1}}$ on $M = \mathbb{S}^1 \times \mathbb{S}^{n-1}$ where $n \geq 3$. Then $f(t, \omega) = \sin(t)$ is a static potential for g . Clearly, scaling the \mathbb{S}^1 -factor preserves the scalar curvature while the total volume changes. Thus g is not a critical point for the volume functional on $\mathcal{M}^{(n-1)(n-2)}$.

To summarize the above discussion, let \mathcal{K} be the space of V -static metrics g on a closed connected manifold M of dimension at least three. This space contains all Einstein metrics and all metrics that are static. By Theorem A in [24], a metric which is Einstein and static is either Ricci-flat or a round sphere. We can write \mathcal{K} as a disjoint union $\mathcal{K} = \mathcal{K}_+ \cup \mathcal{K}_0 \cup \mathcal{K}_-$ according to the sign of the constant scalar curvature $R(g) = c$, cf. Proposition 2.1. By Example 1.3, the space \mathcal{K}_- consists precisely of the Einstein metrics of negative scalar curvature. None of these metrics is static. By Example 1.4, \mathcal{K}_0 is the space of Ricci-flat metrics, all of which are static, and none of which are critical for V_0 . The structure of \mathcal{K}_+ is more complicated. \mathcal{K}_+ consists of metrics that are critical for V_c , e.g. the Einstein metrics of positive scalar curvature, and static metrics that are not critical for V_c , cf. with Example 1.5. Static metrics in \mathcal{K}_+ admit $\frac{c}{n-1}$ in the spectrum of the Laplacian, such as the sphere ($c = n(n-1)$) and $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ ($c = (n-1)(n-2)$). Further examples have been found by Kobayashi and Lafontaine in [29]. If $\frac{c}{n-1}$ is not in the spectrum of the Laplacian of a metric in \mathcal{K}_+ , then the metric is Einstein and non-static, for example \mathbb{RP}^n .

As our first application of Theorems 1.1 and Theorem 1.2, we establish a gluing result which is largely inspired by those in [5, 13, 14]. The result gives a condition that guarantees that two metrics with the same constant scalar curvature can be glued together to produce a metric with the same constant scalar curvature, preserving both the total volume and the original metrics outside a specified region.

Theorem 1.6. Fix $n \geq 3$ and $k \geq 4$. Let $\sigma_n \in \{-n(n-1), 0, n(n-1)\}$. Let (M_1, g_1) and (M_2, g_2) be two compact $C^{k,\alpha}$ Riemannian manifolds such that $R(g_1) = \sigma_n = R(g_2)$. Assume that each (M_i, g_i) contains a non-empty smooth domain $U_i \subset \text{int}(M_i)$ where g_i is not V -static. There exists a $C^{k-2,\alpha}$ metric g

on the connected sum $M_1 \# M_2 \supset (M_1 \setminus U_1) \sqcup (M_2 \setminus U_2)$ such that $R(g) = \sigma_n$, $\text{vol}(M_1 \# M_2, g) = \text{vol}(M_1, g_1) + \text{vol}(M_2, g_2)$, and $g = g_i$ on $M_i \setminus U_i$, $i = 1, 2$. If (M_1, g_1) and (M_2, g_2) are smooth, then we can find $(M_1 \# M_2, g)$ smooth with these properties.

There are many gluing results for constant scalar curvature and, more generally, the Einstein constraint equations in the literature. Gromov-Lawson [12] and Schoen-Yau [27] used different methods to prove that the existence of a positive scalar curvature metric on a manifold is preserved under surgeries of co-dimension at least three. The seminal paper of Schoen [25] on the singular Yamabe problem on the sphere has inspired a large number of works on scalar curvature gluing constructions. The resolution of the Yamabe problem shows that the connected sum of two closed manifolds admits a metric of constant scalar curvature in the conformal class of any metric on the sum. It is interesting and important to understand in what way the constant scalar curvature metric on the sum can be made to reflect the geometry of the original summands. Joyce [15] produced constant scalar curvature metrics on connected sums of closed manifolds by constructing approximate solutions on the joined manifolds by hand, and then solving for a conformal deformation to constant scalar curvature. He also described the geometry of the resulting configuration. A difference in Theorem 1.6 (as in [5, Theorem 1.2]) is that we use a deformation out of the conformal class to preserve the initial metrics away from the gluing region. In particular, we note that in [15], the resulting metric on the connected sum of two zero scalar curvature metrics has constant negative scalar curvature.

The conformal part of the proof of Theorem 1.6 follows closely the works [13, 14], see also [18], on gluing constructions for the Einstein constraint equations. An important observation for localized gluing was made by Chruściel-Delay [4]. They noticed that the conformal constructions could be combined with the localized deformation technique of Corvino-Schoen [7, 9] to produce, under certain non-degeneracy conditions, solutions to the Einstein constraint equations on connected sums for which the original data is left unchanged outside the gluing region. We refer to [5, Theorem 1.2] for an analogue of our Theorem 1.6 in the case $\sigma_n \leq 0$. A gluing construction for constant positive scalar metrics was obtained by Chruściel-Pacard-Pollack [6]. An overview of these constructions with additional references is given in [8, Sections 5.2-5.3]. We also refer the reader to the recent work of Delay [10].

In the final section of this paper, we note how connect-sum constructions for scalar curvature can be combined with the recent counterexample to Min-Oo's conjecture by Brendle, Marques, and Neves [3] to produce counterexamples of different topological types and of large volume. Such examples are interesting in light of the recent results in [22].

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2. VARIATIONAL CHARACTERIZATION OF V -STATIC METRICS

Let $(\bar{\Omega}, g)$ be a connected n -dimensional compact C^3 Riemannian manifold with boundary, and let Ω be the manifold interior of $\bar{\Omega}$. We say that $(\bar{\Omega}, g)$ is V -static

(or simply that the metric g is V -static) if the equation

$$(2.1) \quad \mathcal{S}_g^*(f, a) = 0 \text{ on } \Omega$$

admits a non-trivial weak solution $(f, a) \in H_{\text{loc}}^1(\Omega) \times \mathbb{R}$; f is then called a V -static potential. We will see in Proposition 2.2 that every solution $(f, a) \in H_{\text{loc}}^1(\Omega) \times \mathbb{R}$ of (2.1) is actually in $C^2(\overline{\Omega}) \times \mathbb{R}$. The goal of this section is to study properties of V -static metrics and to characterize the boundary values of V -static potentials.

2.1. The kernel of \mathcal{S}_g^* . The equation

$$(2.2) \quad L_g^* f = \kappa g \text{ on } \Omega$$

is equivalent to (2.1) with $a = -2\kappa$. Given $\kappa \in \mathbb{R}$, $L_g^* f = \kappa g$ is an overdetermined elliptic system for f . It is well-known how to re-cast (2.2) into a proper elliptic system for (f, g) in appropriate coordinates (e.g. harmonic coordinates), cf. [1] or [7, p. 145-146]. In such coordinates, then, f and g are analytic. It follows that if $(\overline{\Omega}, g)$ is V -static, then so is any subdomain (with the restricted metric).

The following property of V -static metrics follows as in [7, Proposition 2.3], see also [20, Theorem 7 (i)].

Proposition 2.1. *Assume that for some constant $\kappa \in \mathbb{R}$ there exists non-trivial a weak solution $f \in H_{\text{loc}}^1(\Omega)$ of (2.2). Then g has constant scalar curvature.*

Proof. By elliptic regularity, $f \in C^2(\Omega) \cap H_{\text{loc}}^3(\Omega)$. Taking the divergence of the equation $L_g^* f = \kappa g$ and using the Bianchi identity and the Ricci formula, it follows that $f dR(g) = 0$. Along a unit-speed geodesic γ with $\gamma(0) = p$, the equation $L_g^* f = \kappa g$ reduces to a second-order ODE with initial data $(f(p), df(\gamma'(0)))$. Indeed, if $h(t) = f(\gamma(t))$, then

$$h''(t) = \nabla_g^2 f(\gamma'(t), \gamma'(t)) = \left(\text{Ric}(g)(\gamma'(t), \gamma'(t)) - \frac{1}{n-1} \right) h(t) - \frac{\kappa}{n-1}.$$

In the homogeneous case $\kappa = 0$, observe that if f has a zero that is a critical point, then f is identically zero. Thus the zero set of f has codimension one. It follows that $dR(g) = 0$ so that the scalar curvature $R(g)$ is constant. If $\kappa \neq 0$, then a solution of the (inhomogeneous) ODE cannot vanish identically in a non-empty open set, from which we can again conclude that $R(g)$ is constant. \square

The ODE argument in the proof of Proposition 2.1 shows that the kernel of L_g^* has dimension at most $(n+1)$. Thus, the dimension of the kernel of \mathcal{S}_g^* is at most $(n+2)$. This maximal dimension is achieved, for example, by the standard metric on the sphere \mathbb{S}^n . Viewing \mathbb{S}^n as the unit sphere in \mathbb{R}^{n+1} with center at the origin, the kernel is spanned by $(x^j|_{\mathbb{S}^n}, 0)$, $j = 1, \dots, n+1$, and $(1, 2(n-1))$ in this case.

By employing the exponential map from points near the boundary and using basic facts about existence, uniqueness, and dependence on initial data for ODEs as in [7, Proposition 2.5], we see that every solution f of (2.2) extends to the boundary as a C^2 function; using an interior elliptic estimate, or appealing to the finite-dimensionality of the kernel, we also obtain an estimate on such solutions:

Proposition 2.2. *Every weak solution $f \in H_{\text{loc}}^1(\Omega)$ of (2.2) is actually in $C^2(\overline{\Omega})$. There is a constant $C = C(\Omega, g)$ so that for $\epsilon > 0$ sufficiently small, and for any solution f of (2.2), $\|f\|_{C^2(\overline{\Omega})} \leq C\|f\|_{H^1(\Omega_\epsilon)}$, where $\Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) > \epsilon\}$.*

2.2. The variational principle. In this section, we characterize the boundary values of solutions f of (2.2) whose existence is ensured by Proposition 2.2. For simplicity, we will assume that $n \geq 3$ and that $(\overline{\Omega}, g)$ is smooth in this subsection and the next.

We adopt the notation from [20]. Let γ be a smooth Riemannian metric on $\partial\Omega$. Let c be a constant. For any integer $k > \frac{n}{2} + 2$, let \mathcal{M}_γ^c denote the set of H^k Riemannian metrics g on $\overline{\Omega}$ such that $R(g) = c$ and $g|_{T(\partial\Omega)} = \gamma$, where $R(g)$ is the scalar curvature of g and $g|_{T(\partial\Omega)}$ is the metric induced by g on $\partial\Omega$. We recall from [20] that if g is such that $\Delta_g + \frac{c}{n-1}$ has positive (Dirichlet) spectrum, then \mathcal{M}_γ^c is a Hilbert manifold near g . Let ν be the outward unit normal to $\partial\Omega$, let $\mathbb{I}\!\!\mathbb{I}(X, Y) = \langle \nabla_X \nu, Y \rangle$ for vector fields X, Y tangent to $\partial\Omega$, and let $H = \text{tr}_\gamma(\mathbb{I}\!\!\mathbb{I})$ be the mean curvature. (Our sign convention follows that of [20].)

The following theorem provides a general context unifying [20, Theorem 5] and [19, Theorem 2.1].

Theorem 2.3. *Let κ be a constant and let ϕ be a smooth function on $\partial\Omega$. We assume that either $\kappa \neq 0$ or that ϕ does not vanish identically. Consider the functional on \mathcal{M}_γ^c given by*

$$(2.3) \quad g \mapsto E_{\kappa, \phi}(g) = \kappa V(g) - \int_{\partial\Omega} H\phi \, d\sigma,$$

where $V(g)$ is the volume of $(\overline{\Omega}, g)$ and $d\sigma$ is the volume form of γ . Suppose $g \in \mathcal{M}_\gamma^c$ is a smooth metric such that the operator $\Delta_g + \frac{c}{n-1}$ has positive (Dirichlet) spectrum. Then g is a critical point of $E_{\kappa, \phi}(\cdot)$ on \mathcal{M}_γ^c if and only if there exists a smooth function f on $\overline{\Omega}$ with

$$(2.4) \quad L_g^* f = \kappa g \text{ in } \Omega \text{ and } f = \phi \text{ on } \partial\Omega.$$

Proof. We follow the proof of Theorem 2.1 in [19]. Let $\{g(t)\}_{|t| \leq \epsilon} \subset \mathcal{M}_\gamma^c$ be a continuously differentiable path such that $g(0) = g$. Let $h = g'(0)$. Let $H(t)$ be the mean curvature of $\partial\Omega$ in $(\overline{\Omega}, g(t))$ computed with respect to the outward unit normal as above. A calculation as in [20, (34)] yields that

$$(2.5) \quad 2H'(0) = [d(\text{tr}_g h) - \text{div}_g h](\nu) - \text{div}_\gamma X - \langle \mathbb{I}\!\!\mathbb{I}, h \rangle_\gamma$$

where ν is the outward unit normal to $\partial\Omega$ in $(\Omega, g(0))$, X is the vector field dual to the 1-form $h(\nu, \cdot)|_{T(\partial\Omega)}$ on $(\partial\Omega, \gamma)$, $\text{div}_\gamma X$ is the divergence of X on $(\partial\Omega, \gamma)$, and $\langle \cdot, \cdot \rangle_\gamma$ is the metric product on $(\partial\Omega, \gamma)$. Using that $h|_{T(\partial\Omega)} = 0$, it follows that

$$(2.6) \quad 2 \frac{d}{dt} \Big|_{t=0} E_{\kappa, \phi}(g(t)) = \int_\Omega \kappa \text{tr}_g h - \int_{\partial\Omega} \phi \{ [d(\text{tr}_g h) - \text{div}_g h](\nu) - \text{div}_\gamma X \}$$

where we have omitted the volume forms. For any function f on Ω with $f = \phi$ along $\partial\Omega$, we can integrate by parts in (2.6) to obtain

$$(2.7) \quad \begin{aligned} 2 \frac{d}{dt} \Big|_{t=0} E_{\kappa, \phi}(g(t)) &= \int_\Omega \kappa \text{tr}_g h - f [\Delta_g(\text{tr}_g h) - \text{div}_g(\text{div}_g h)] + (\Delta_g f) \text{tr}_g h - \langle \nabla_g^2 f, h \rangle_g \\ &\quad + \int_{\partial\Omega} h(\nu, \nabla_g f) - h(\nu, \nabla_\gamma f) - \text{tr}_g h \frac{\partial f}{\partial \nu} \end{aligned}$$

where $\langle \cdot, \cdot \rangle_g$ denotes the metric product on $(\bar{\Omega}, g)$ and ∇_γ is the gradient operator on $(\partial\Omega, \gamma)$. Since $h|_{T(\partial\Omega)} = 0$,

$$(2.8) \quad h(\nu, \nabla_g f) - h(\nu, \nabla_\gamma f) - \text{tr}_g h \frac{\partial f}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

On the other hand, the fact that $\{g(t)\}_{|t| \leq \epsilon} \subset \mathcal{M}_\gamma^c$ implies that

$$(2.9) \quad L_g(h) = -\Delta_g(\text{tr}_g h) + \text{div}_g \text{div}_g h - \langle h, \text{Ric}(g) \rangle_g = 0,$$

where we recall that $L_g(h) = DR_g(h)$ is the linearization of the scalar curvature map at g in direction h . Therefore, it follows from (2.7)-(2.9) that

$$(2.10) \quad \begin{aligned} 2 \frac{d}{dt} \Big|_{t=0} E_{\kappa, \phi}(g(t)) &= \int_{\Omega} \langle h, f \text{Ric}(g) + (\Delta_g f)g - \nabla_g^2 f + \kappa g \rangle_g \\ &= \int_{\Omega} \langle h, -L_g^* f + \kappa g \rangle_g = 0. \end{aligned}$$

Hence if f is a solution of (2.4), then g is a critical point of $E_{\kappa, \phi}(\cdot)$ on \mathcal{M}_γ^c .

For the other direction, assume now that g is a critical point of $E_{\kappa, \phi}(\cdot)$, and consider the unique solution f of the boundary value problem

$$(2.11) \quad \begin{cases} (n-1)\Delta_g f + cf &= -n\kappa & \text{in } \Omega \\ f &= \phi & \text{on } \partial\Omega. \end{cases}$$

Let \hat{h} be an arbitrary smooth symmetric $(0, 2)$ -tensor with compact support in Ω . Since the first Dirichlet eigenvalue of $\Delta_g + \frac{c}{n-1}$ is positive, by [20, Proposition 1] there exist $t_0 > 0$ and $\epsilon > 0$ such that, for every $t \in (-t_0, t_0)$, there exists a unique smooth positive function $u(t)$ on $\bar{\Omega}$ with $|u(t) - 1| \leq \epsilon$ such that $u(t) = 1$ on $\partial\Omega$, such that $g(t) = u(t)^{\frac{4}{n-2}}(g + t\hat{h}) \in \mathcal{M}_\gamma^c$, and such that $\{u(t)\}_{|t| < t_0}$ is differentiable at $t = 0$ with $u(0) = 1$. For such a path $g(t)$, we have $h := g'(0) = \frac{4}{n-2}u'(0)g + \hat{h}$. Hence, by (2.10) and the fact that f is a solution to (2.11), we have

$$(2.12) \quad 0 = \int_{\Omega} \langle \hat{h}, f \text{Ric}(g) + (\Delta_g f)g - \nabla_g^2 f + \kappa g \rangle_g.$$

Since \hat{h} can be chosen arbitrarily, we conclude that f satisfies (2.4). \square

2.3. A volume comparison result for V -static metrics. When the function ϕ in Theorem 2.3 is chosen to be identically zero and $\kappa = 1$, then Theorem 2.3 reduces to Theorem 5 in [20] and claims that for a metric $g \in \mathcal{M}_\gamma^c$ for which $\Delta_g + \frac{c}{n-1}$ has positive first Dirichlet eigenvalue, the system

$$(2.13) \quad \begin{cases} -(\Delta_g f)g + \nabla_g^2 f - f \text{Ric}(g) &= g & \text{in } \Omega \\ f &= 0 & \text{on } \partial\Omega \end{cases}$$

admits a solution $f \in C^2(\bar{\Omega})$ if and only if g is a critical point of the volume functional $V(\cdot)$ restricted to \mathcal{M}_γ^c . We recall the following volume comparison result from [20] for such metrics when $c = 0$.

Theorem 2.4 ([20]). *Let g be a smooth, scalar flat metric on $\bar{\Omega}$. Suppose there exists a function f such that g and f satisfy (2.13). Let γ be the metric induced on $\Sigma = \partial\Omega$. Suppose Σ is connected and that (Σ, γ) can be isometrically embedded in \mathbb{R}^n as a compact strictly convex hypersurface Σ_0 . If $n > 7$, where n is the dimension of $\bar{\Omega}$, we assume in addition that $\bar{\Omega}$ is spin. Then*

$$V(g) \geq V_0$$

where $V(g)$ is the volume of $(\bar{\Omega}, g)$ and V_0 is the Euclidean volume of the compact domain bounded by Σ_0 in \mathbb{R}^n . Moreover, $V(g) = V_0$ if and only if $(\bar{\Omega}, g)$ is isometric to a standard ball in \mathbb{R}^n .

The proof of Theorem 2.4 in [20] uses the result of Shi and Tam in [30] and thus depends on the Positive Mass Theorem [26, 32]. Here we include another proof of Theorem 2.4 that does not depend on the Positive Mass Theorem, so we can omit the spin assumption in high dimensions. We start with the following proposition.

Proposition 2.5. *Let g be a smooth, scalar flat metric on $\bar{\Omega}$. Suppose there exists a function f such that g and f satisfy (2.13). Let γ be the metric induced on $\Sigma = \partial\Omega$, let $|\Sigma|$ be the area of (Σ, γ) , and suppose that Σ is connected. Then*

- a) $\int_{\Sigma} R_{\gamma} > 0$, where R_{γ} is the scalar curvature of (Σ, γ) .
- b) The volume $V(g)$ of $(\bar{\Omega}, g)$ satisfies

$$V(g) \geq \frac{\sqrt{(n-2)(n-1)}}{n} \left(\int_{\Sigma} R_{\gamma} \right)^{-\frac{1}{2}} |\Sigma|^{\frac{3}{2}}.$$

Equality holds if and only if $(\bar{\Omega}, g)$ is isometric to a standard ball in \mathbb{R}^n .

- c) When $n = 3$, one has

$$V(g) \geq \frac{|\Sigma|^{\frac{3}{2}}}{6\sqrt{\pi}}.$$

Equality holds if and only if $(\bar{\Omega}, g)$ is isometric to a round ball in \mathbb{R}^3 .

Proof. Let ν be the outward unit normal to Σ . Let H and $\mathbb{I}\mathbb{I}$ be the mean curvature and the second fundamental form of Σ in $(\bar{\Omega}, g)$ with respect to ν . By Theorem 7 (iii) in [20], f is positive on Ω , H is a positive constant, and $(n-1)\mathbb{I}\mathbb{I} = H\gamma$. Moreover, by (48) and (53) in [20], we have that $H \frac{\partial f}{\partial \nu} = -1$ and $|\Sigma| = \frac{n}{n-1} H V(g)$. Hence

(2.14)

$$\int_{\Omega} \langle \nabla_g^2 f, \text{Ric}(g) \rangle = \int_{\Sigma} \text{Ric}(g)(\nu, \nabla_g f) - \int_{\Omega} \langle df, \text{div}_g \text{Ric}(g) \rangle = \frac{\partial f}{\partial \nu} \int_{\Sigma} \text{Ric}(g)(\nu, \nu)$$

where the first equality follows from an integration by parts, and where we used that $2\text{div}_g(\text{Ric}(g)) = dR(g) = 0$ and that $\frac{\partial f}{\partial \nu}$ is constant along Σ to justify the second equality. Taking the metric product of (2.13) with $\text{Ric}(g)$ and using again that $R(g) = 0$, we see that

$$(2.15) \quad \langle \nabla_g^2 f, \text{Ric}(g) \rangle = \langle f \text{Ric}(g) + (\Delta_g f)g + g, \text{Ric}(g) \rangle = f |\text{Ric}(g)|^2.$$

Equations (2.14) and (2.15), together with the fact that $H \frac{\partial f}{\partial \nu} = -1$, give

$$(2.16) \quad - \int_{\Omega} f |\text{Ric}(g)|^2 = \frac{1}{H} \int_{\Sigma} \text{Ric}(g)(\nu, \nu).$$

In particular, this shows that

$$(2.17) \quad \int_{\Sigma} \text{Ric}(g)(\nu, \nu) \leq 0,$$

with equality if and only if $\text{Ric}(g) = 0$ on Ω .

The Gauss Equation, along with the fact $R(g) = 0$ and $\mathbb{I}\mathbb{I} = \frac{H}{n-1}\gamma$, implies

$$(2.18) \quad 2\text{Ric}(g)(\nu, \nu) = \frac{n-2}{n-1} H^2 - R_{\gamma}.$$

It follows from (2.17) and (2.18) that

$$(2.19) \quad \int_{\Sigma} R_{\gamma} \geq \frac{n-2}{n-1} \int_{\Sigma} H^2 = \frac{n-2}{n-1} H^2 |\Sigma|.$$

This proves a).

The inequality in b) follows from (2.19) and the fact $|\Sigma| = \frac{n}{n-1} HV(g)$. If equality holds, then $\text{Ric}(g) = 0$ on Ω . That $(\overline{\Omega}, g)$ is isometric to a ball in \mathbb{R}^n in this case then follows from [21, Theorem 2.1].

Finally, c) follows from b) and the Gauss-Bonnet Theorem. \square

The fact that Proposition 2.5 implies Theorem 2.4 was first noted by Tam [31]. We thank Luen-Fai Tam for pointing out the following lemma.

Lemma 2.6. *Suppose $\Sigma \subset \mathbb{R}^n$ is an embedded, closed, strictly convex hypersurface. Let R_{γ} be the scalar curvature of Σ with respect to the metric γ induced from the Euclidean metric, let $|\Sigma|$ be its area, and let V be the Euclidean volume of the region enclosed by Σ . Then*

$$(2.20) \quad \int_{\Sigma} R_{\gamma} \leq \frac{(n-1)(n-2)|\Sigma|^3}{n^2 V^2}.$$

Proof. Let H denote the (positive) mean curvature of Σ . Taking i, j, k to be 1, 2, 3 and then 0, 1, 2 in (6.4.6) in [28, p. 334], one arrives at two of the Minkowski inequalities

$$(2.21) \quad (W_1)^2 \geq W_0 W_2, \quad (W_2)^2 \geq W_1 W_3,$$

where $W_0 = V$, $W_1 = \frac{1}{n} |\Sigma|$, $W_2 = \frac{1}{n(n-1)} \int_{\Sigma} H$, and $W_3 = \frac{1}{n(n-1)(n-2)} \int_{\Sigma} R_{\gamma}$. Clearly, (2.21) implies (2.20). \square

Theorem 2.4 without the spin assumption in dimension $n > 7$ now follows from Proposition 2.5 and Lemma 2.6.

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is similar to those of the localized deformation theorems in [7, 9, 4]. It proceeds by iteration with a linear correction at each stage. The linearized problem is solved variationally. This requires delicate weighted L^2 -estimates. The pointwise bounds on these variational solutions required to establish convergence of the iteration follow from interior Schauder estimates.

3.1. Function spaces. Let k be a non-negative integer, $\alpha \in (0, 1)$, and let $(\overline{\Omega}, g)$ be a compact $C^{k, \alpha}$ Riemannian manifold with boundary. Let Ω denote the manifold interior of $\overline{\Omega}$. Let $\ell \leq k$ be a non-negative integer, and let ρ be a positive measurable function on Ω . Below, we use the connection and the tensor norms induced by g , and we integrate with respect to the volume form $d\mu_g$.

Let $L_{\rho}^2(\Omega)$ be the set of functions (or tensor fields) u such that $|u|\rho^{1/2} \in L^2(\Omega)$ and let $\|u\|_{L_{\rho}^2(\Omega)} = \|u\rho^{1/2}\|_{L^2(\Omega)}$. The pairing

$$\langle u, v \rangle_{L_{\rho}^2(\Omega)} = \langle u\rho^{1/2}, v\rho^{1/2} \rangle_{L^2(\Omega)}$$

makes $L_{\rho}^2(\Omega)$ into a Hilbert space. Let $H_{\rho}^{\ell}(\Omega)$ be the Hilbert space of $L_{\rho}^2(\Omega)$ functions (tensor fields) whose covariant derivatives up to and including order ℓ are also

in $L^2_\rho(\Omega)$. The inner product is defined by incorporating the $L^2_\rho(\Omega)$ -pairings on all the derivatives, so that

$$\|u\|_{H^\ell_\rho(\Omega)}^2 = \sum_{j=0}^{\ell} \|\nabla_g^j u\|_{L^2_\rho(\Omega)}^2.$$

Assume now that $k \geq 1$. Let $d(x) = d(x, \partial\Omega)$ be the distance to the boundary $\partial\Omega$ computed with respect to the metric g . Then $d(x)$ is a $C^{k,\alpha}$ function near $\partial\Omega$ [11]. We will use a $C^{k,\alpha}$ weight ρ with $0 < \rho \leq 1$ on Ω with the following boundary behavior: ρ depends monotonically on the distance d to $\partial\Omega$, $\rho = e^{-1/d}$ near $\partial\Omega$, and $\rho \equiv 1$ outside a neighborhood of $\partial\Omega$. Note that $|\nabla_g^j \rho| \leq C(j)d^{-2j}e^{-1/d}$. To be precise, we let $\rho(x) = \tilde{\rho}(d(x))$, where $\tilde{\rho} : \mathbb{R} \rightarrow [0, 1]$ is smooth and monotone, $\tilde{\rho}' \geq 0$, with $\tilde{\rho}(t) > 0$ for $t > 0$, and $\tilde{\rho}(t) = e^{-1/t}$ on some interval $(0, d_0)$.

Let $\phi > 0$ be a $C^{k,\alpha}$ function on Ω such that for all $x \in \Omega$, $B(x, \phi(x)) \subset \Omega$, and so that near $\partial\Omega$, $\phi = d^2$. For $r, s \in \mathbb{R}$, let $\varphi = \phi^r \rho^s$. For a $C^{\ell,\alpha}$ function $u : \Omega \rightarrow \mathbb{R}$ we define $\|u\|_{C^{\ell,\alpha}_{\phi,\varphi}(\Omega)}$ by

$$\sup_{x \in \Omega} \left(\sum_{j=0}^{\ell} \varphi(x) \phi(x)^j \|\nabla_g^j u\|_{C^0(B(x, \phi(x)/2))} + \varphi(x) \phi(x)^{\ell+\alpha} [\nabla_g^\ell u]_{0,\alpha;B(x, \phi(x)/2)} \right).$$

We let $C^{\ell,\alpha}_{\phi,\varphi}(\Omega)$ be the space of all functions $u \in C^{\ell,\alpha}(\Omega)$ for which $\|u\|_{C^{\ell,\alpha}_{\phi,\varphi}(\Omega)} < \infty$. Note that $\|\cdot\|_{C^{\ell,\alpha}_{\phi,\varphi}(\Omega)}$ is a Banach norm on this space. When the context is clear, we will suppress the domain in the notation below. With our choice of ϕ and φ , we have that $\|u\|_{C^{\ell,\alpha}_{\phi,\varphi}}$ is equivalent to $\|u\varphi\|_{C^{\ell,\alpha}_{\phi,1}}$. Moreover, differentiation is continuous as a map from $C^{\ell,\alpha}_{\phi,\varphi}(\Omega)$ to $C^{\ell-1,\alpha}_{\phi,\varphi}(\Omega)$.

These weighted Hölder norms are equivalent to those defined in [4, p. 66].

3.2. Coercivity estimate for \mathcal{S}_g^* . Let $\{x \in \Omega : d(x) < \epsilon_0\}$ be a regular tubular neighborhood of $\partial\Omega$ and let $\Omega_\epsilon = \{x \in \Omega : d(x) > \epsilon\}$.

Proposition 3.1 (Cf. [7, Theorem 3]). *Let $(\overline{\Omega}, g)$ be as in Theorem 1.1. There exists a constant $C > 0$ so that for all $(u, a) \in H^2_\rho(\Omega) \times \mathbb{R}$,*

$$(3.1) \quad \|(u, a)\|_{H^2_\rho(\Omega) \times \mathbb{R}} \leq C \|\mathcal{S}_g^*(u, a)\|_{L^2_\rho(\Omega)}.$$

Proof. There is a constant $D > 0$ so that for all $\epsilon > 0$ sufficiently small, there is an extension operator $E_\epsilon : H^2(\Omega_\epsilon) \rightarrow H^2(\Omega)$ with norm bounded by D . The equation $\mathcal{S}_g^*(u, a) = -(\Delta_g u)g + \nabla_g^2 u - u \text{Ric}(g) + \frac{a}{2}g$ shows that

$$(3.2) \quad \|(u, a)\|_{H^2(\Omega_\epsilon) \times \mathbb{R}} \leq C(n, g, \Omega) \left(\|\mathcal{S}_g^*(u, a)\|_{L^2(\Omega_\epsilon)} + \|(u, a)\|_{H^1(\Omega_\epsilon) \times \mathbb{R}} \right).$$

Note that \mathcal{S}_g^* has trivial kernel in $H^1_{\text{loc}}(\Omega_\epsilon)$ for $\epsilon > 0$ sufficiently small. Indeed, the kernels K_ϵ of \mathcal{S}_g^* on Ω_ϵ decrease as $\epsilon \downarrow 0$ (by restriction, which is injective by the remarks following (2.2)). Since each is at most $(n+2)$ -dimensional (cf. Section 2.1) and there is no kernel on Ω , they must stabilize at $\{0\}$.

We claim that there is a constant $C > 0$ so that for all $\epsilon > 0$ sufficiently small and $(u, a) \in H^2(\Omega_\epsilon) \times \mathbb{R}$,

$$(3.3) \quad \|(u, a)\|_{H^2(\Omega_\epsilon) \times \mathbb{R}} \leq C \|\mathcal{S}_g^*(u, a)\|_{L^2(\Omega_\epsilon)}.$$

We prove this by contradiction. Suppose the estimate does not hold. There is a sequence $\epsilon_j > 0$ with $\epsilon_j \downarrow 0$ and $(u_j, a_j) \in H^2(\Omega_{\epsilon_j}) \times \mathbb{R}$ such that

$$(3.4) \quad \|(u_j, a_j)\|_{H^2(\Omega_{\epsilon_j}) \times \mathbb{R}} \geq j \|\mathcal{S}_g^*(u_j, a_j)\|_{L^2(\Omega_{\epsilon_j})}.$$

Let $\tilde{u}_j = E_{\epsilon_j}(u_j)$ be the extension of u_j to Ω . Then

$$(3.5) \quad \|(u_j, a_j)\|_{H^2(\Omega_{\epsilon_j}) \times \mathbb{R}} \leq \|(\tilde{u}_j, a_j)\|_{H^2(\Omega) \times \mathbb{R}} \leq D \|(u_j, a_j)\|_{H^2(\Omega_{\epsilon_j}) \times \mathbb{R}}.$$

We normalize so that $\|(\tilde{u}_j, a_j)\|_{H^1(\Omega) \times \mathbb{R}} = 1$. Using (3.2) and (3.4), we obtain

$$\begin{aligned} \|(u_j, a_j)\|_{H^2(\Omega_{\epsilon_j}) \times \mathbb{R}} &\leq C(n, g, \Omega) \left(\|\mathcal{S}_g^*(u_j, a_j)\|_{L^2(\Omega_{\epsilon_j})} + \|(u_j, a_j)\|_{H^1(\Omega_{\epsilon_j}) \times \mathbb{R}} \right) \\ &\leq C(n, g, \Omega) \left(j^{-1} \|(u_j, a_j)\|_{H^2(\Omega_{\epsilon_j}) \times \mathbb{R}} + 1 \right). \end{aligned}$$

For j large enough so that $C(n, g, \Omega)j^{-1} \leq \frac{1}{2}$, we obtain $\|(u_j, a_j)\|_{H^2(\Omega_{\epsilon_j}) \times \mathbb{R}} \leq 2C(n, g, \Omega)$. By (3.5), we then have $\|(\tilde{u}_j, a_j)\|_{H^2(\Omega) \times \mathbb{R}} \leq 2DC(n, g, \Omega)$. By the Rellich Lemma and the fact that the sequence $\{a_j\}$ is bounded, there exist $(u, a) \in H^2(\Omega) \times \mathbb{R}$ and a subsequence of $\{(\tilde{u}_j, a_j)\}$ that converges to (u, a) weakly in $H^2(\Omega) \times \mathbb{R}$ and strongly in $H^1(\Omega) \times \mathbb{R}$. The latter implies that $\|(u, a)\|_{H^1(\Omega) \times \mathbb{R}} = 1$. Moreover, by pairing $\mathcal{S}_g^*(u, a)$ in $L^2(\Omega)$ with $h \in C_c^2(\Omega)$, and using (3.4), we see $\mathcal{S}_g^*(u, a) = 0$ holds weakly, so that (u, a) is a non-trivial element of the kernel of \mathcal{S}_g^* . This is a contradiction. Thus (3.3) holds uniformly for $\epsilon > 0$ small, as asserted.

The uniformity of (3.3) in $\epsilon > 0$ allows us to promote this estimate to the weighted coercivity estimate (3.1) exactly as in [7, p. 149-150], using the co-area formula and integration by parts. Indeed, for any $u \in C^2(\overline{\Omega})$, and any sufficiently small $d_1 > 0$, we have that

$$\int_0^{d_1} \rho'(\epsilon) \|u\|_{H^2(\Omega_\epsilon \setminus \overline{\Omega_{d_1}})}^2 d\epsilon = \|u\|_{H_\rho^2(\Omega \setminus \overline{\Omega_{d_1}})}^2.$$

With $C_0 = \rho(d_1) = \int_0^{d_1} \rho'(\epsilon) d\epsilon > 0$ and (3.3), this implies

$$\begin{aligned} C_0(\|u\|_{H^2(\Omega_{d_1})}^2 + a^2) + \|u\|_{H_\rho^2(\Omega \setminus \overline{\Omega_{d_1}})}^2 &\leq C^2 \int_0^{d_1} \rho'(\epsilon) \|\mathcal{S}_g^*(u, a)\|_{L^2(\Omega_\epsilon)}^2 d\epsilon \\ &\leq C^2 C_0 \|\mathcal{S}_g^*(u, a)\|_{L^2(\Omega_{d_1})}^2 + C^2 \|\mathcal{S}_g^*(u, a)\|_{L_\rho^2(\Omega \setminus \overline{\Omega_{d_1}})}^2. \end{aligned}$$

By the density of $C^2(\overline{\Omega})$ in $H_\rho^2(\Omega)$ (cf. [9, Lemma 2.1]), (3.1) now follows easily. \square

3.3. Variational solution of the linearized equation. Under the assumption that \mathcal{S}_g^* has trivial kernel, solutions to $\mathcal{S}_g(h) = (\sigma, \tau)$ for $(\sigma, \tau) \in L_{\rho^{-1}}^2(\Omega) \times \mathbb{R}$ can be obtained from a standard variational argument.

Proposition 3.2 (Cf. [7, Proposition 3.6]). *Let $(\overline{\Omega}, g)$ be as in Theorem 1.1. Let $(\sigma, \tau) \in L_{\rho^{-1}}^2(\Omega) \times \mathbb{R}$. Define the functional $\mathcal{F} : H_\rho^2(\Omega) \times \mathbb{R} \rightarrow \mathbb{R}$ by*

$$(3.6) \quad \mathcal{F}(u, a) = \int_\Omega \left(\frac{1}{2} |\mathcal{S}_g^*(u, a)|^2 \rho - \sigma u \right) d\mu_g - a\tau.$$

Then \mathcal{F} has a unique critical point $(u, a) \in H_\rho^2(\Omega) \times \mathbb{R}$. This critical point is the global minimizer of \mathcal{F} and it is a weak solution of the equation $\mathcal{S}_g(\rho \mathcal{S}_g^(u, a)) =$*

(σ, τ) . There is a constant $C > 0$ such that for every $(\sigma, \tau) \in L^2_{\rho^{-1}}(\Omega) \times \mathbb{R}$ the minimizer $(u, a) \in H^2_\rho(\Omega) \times \mathbb{R}$ of the corresponding functional satisfies $\|(u, a)\|_{H^2_\rho(\Omega) \times \mathbb{R}} \leq C\|(\sigma, \tau)\|_{L^2_{\rho^{-1}}(\Omega) \times \mathbb{R}}$.

Proof. Let $\mu = \inf\{\mathcal{F}(u, a) : (u, a) \in H^2_\rho(\Omega) \times \mathbb{R}\}$. The choice $(u, a) = (0, 0)$ shows that $\mu \leq 0$. The coercivity estimate (3.1) shows that μ is finite. Standard Hilbert space arguments exactly as in [7, p. 150-152] show that a minimizer $(u, a) \in H^2_\rho(\Omega) \times \mathbb{R}$ of \mathcal{F} exists.

If $(u, a) \neq (\hat{u}, \hat{a}) \in H^2_\rho(\Omega) \times \mathbb{R}$, then $\mathcal{S}_g^*(u - \hat{u}, a - \hat{a}) \neq 0$, and the map $t \mapsto \mathcal{F}((1-t)(u, a) + t(\hat{u}, \hat{a}))$ is strictly convex. This shows that (u, a) is the unique critical point and in particular the only global minimizer of \mathcal{F} .

The Euler-Lagrange condition for the critical point (u, a) of \mathcal{F} gives that for all $(v, b) \in C_c^2(\Omega) \times \mathbb{R}$,

$$\int_{\Omega} \mathcal{S}_g^*(u, a) \cdot \mathcal{S}_g^*(v, b) \rho \, d\mu_g = \int_{\Omega} \sigma v \, d\mu_g + \tau b.$$

Thus (u, a) is a weak solution of $\mathcal{S}_g(\rho \mathcal{S}_g^*(u, a)) = (\sigma, \tau)$.

Finally, using the coercivity estimate (3.1), Cauchy-Schwarz, and $\mu \leq 0$, we obtain that

$$\begin{aligned} \frac{1}{2C} \|(u, a)\|_{H^2_\rho(\Omega) \times \mathbb{R}}^2 &\leq \int_{\Omega} \frac{1}{2} |\mathcal{S}_g^*(u, a)|^2 \rho \, d\mu_g \\ &= \mu + \int_{\Omega} \sigma u \, d\mu_g + a\tau \\ &\leq \|(\sigma, \tau)\|_{L^2_{\rho^{-1}}(\Omega) \times \mathbb{R}} \cdot \|(u, a)\|_{H^2_\rho(\Omega) \times \mathbb{R}}. \end{aligned}$$

□

3.4. Pointwise estimates of the variational solution. We will use the following function spaces:

$$\begin{aligned} \mathcal{B}_0 &:= \left(C_{\phi, \phi^{4+\frac{n}{2}\rho^{-\frac{1}{2}}}}^{0, \alpha}(\Omega) \cap L_{\rho^{-1}}^2(\Omega) \right) \times \mathbb{R} \\ \mathcal{B}_2 &:= C_{\phi, \phi^{2+\frac{n}{2}\rho^{-\frac{1}{2}}}}^{2, \alpha}(\text{Sym}^2(T^*\Omega)) \cap L_{\rho^{-1}}^2(\text{Sym}^2(T^*\Omega)) \\ \mathcal{B}_4 &:= \left(C_{\phi, \phi^{\frac{n}{2}\rho^{\frac{1}{2}}}}^{4, \alpha}(\Omega) \cap H_{\rho}^2(\Omega) \right) \times \mathbb{R} \end{aligned}$$

with Banach norms

$$\begin{aligned} \|(\sigma, \tau)\|_0 &:= |\tau| + \|\sigma\|_{L_{\rho^{-1}}^2} + \|\sigma\|_{C_{\phi, \phi^{4+\frac{n}{2}\rho^{-\frac{1}{2}}}}^{0, \alpha}} \\ \|h\|_2 &:= \|h\|_{L_{\rho^{-1}}^2} + \|h\|_{C_{\phi, \phi^{2+\frac{n}{2}\rho^{-\frac{1}{2}}}}^{2, \alpha}} \\ \|(u, a)\|_4 &:= |a| + \|u\|_{H_{\rho}^2} + \|u\|_{C_{\phi, \phi^{\frac{n}{2}\rho^{\frac{1}{2}}}}^{4, \alpha}}. \end{aligned}$$

The operator $\rho \mathcal{S}_g^*$ is continuous from $C_{\phi, \phi^{\frac{n}{2}\rho^{\frac{1}{2}}}}^{4, \alpha}(\Omega) \times \mathbb{R}$ to the space of $C_{\phi, \phi^{2+\frac{n}{2}\rho^{-\frac{1}{2}}}}^{2, \alpha}(\Omega)$ sections of $\text{Sym}^2(T^*\Omega)$. The operator \mathcal{S}_g is continuous from the space of $C_{\phi, \phi^{2+\frac{n}{2}\rho^{-\frac{1}{2}}}}^{2, \alpha}(\Omega)$ sections of $\text{Sym}^2(T^*\Omega)$ to $C_{\phi, \phi^{4+\frac{n}{2}\rho^{-\frac{1}{2}}}}^{0, \alpha}(\Omega) \times \mathbb{R}$.

Proposition 3.3. *Let $(\bar{\Omega}, g)$ be as in Theorem 1.1. There exists a constant $C > 0$ with the following property. Given $(\sigma, \tau) \in \mathcal{B}_0$, there is $(u, a) \in \mathcal{B}_4$ so that $\mathcal{S}_g(h) = (\sigma, \tau)$, $\|(u, a)\|_4 \leq C\|(\sigma, \tau)\|_0$, and $\|h\|_2 \leq C\|(\sigma, \tau)\|_0$ where $h = \rho\mathcal{S}_g^*(u, a)$.*

Proof. Fix $(\sigma, \tau) \in \mathcal{B}_0$. Let $(u, a) \in H_\rho^2(\Omega) \times \mathbb{R}$ be the weak solution of $\mathcal{S}_g \rho \mathcal{S}_g^*(u, a) = (\sigma, \tau)$ from Proposition 3.2. Let $h = \rho \mathcal{S}_g^*(u, a) \in L_{\rho^{-1}}^2(\Omega)$. Elliptic regularity for the operator $\rho^{-1} L_g \rho L_g^*$ gives that $h \in C^{2,\alpha}(\Omega)$.

Note that $\rho^{-1} L_g(\rho L_g^* u) = \rho^{-1} \sigma - \frac{a}{2} \rho^{-1} L_g(\rho g)$. We apply the Schauder interior estimates in the form discussed in Appendix A. We also use the bound $\|(u, a)\|_{H_\rho^2 \times \mathbb{R}} \leq C\|(\sigma, \tau)\|_0$ from Proposition 3.2 and the obvious estimate $\|h\|_{L_{\rho^{-1}}^2} \leq C\|(u, a)\|_{H_\rho^2 \times \mathbb{R}}$. The constant C may change from line to line.

$$\begin{aligned}
\|h\|_{C_{\phi, \phi^{2+\frac{n}{2}} \rho^{-1/2}}^{2,\alpha}} &= \|\rho \mathcal{S}_g^*(u, a)\|_{C_{\phi, \phi^{2+\frac{n}{2}} \rho^{-1/2}}^{2,\alpha}} \leq C\|(u, a)\|_{C_{\phi, \phi^{n/2} \rho^{1/2}}^{4,\alpha} \times \mathbb{R}} \\
&\leq C\left(\|\rho^{-1} \sigma - \frac{a}{2} \rho^{-1} L_g(\rho g)\|_{C_{\phi, \phi^{4+\frac{n}{2}} \rho^{1/2}}^{0,\alpha}} + \|(u, a)\|_{L_\rho^2 \times \mathbb{R}}\right) \\
&\leq C\left(\|\rho^{-1} \sigma\|_{C_{\phi, \phi^{4+\frac{n}{2}} \rho^{1/2}}^{0,\alpha}} + \|(u, a)\|_{L_\rho^2 \times \mathbb{R}}\right) \\
&\leq C\left(\|\sigma\|_{C_{\phi, \phi^{4+\frac{n}{2}} \rho^{-1/2}}^{0,\alpha}} + \|(\sigma, \tau)\|_{L_{\rho^{-1}}^2 \times \mathbb{R}}\right) \\
&= C\|(\sigma, \tau)\|_0.
\end{aligned}$$

□

3.5. Solving the non-linear problem by iteration. The goal of this section is to obtain a solution of the non-linear problem $\Theta(g+h) = \Theta(g) + (\sigma, \tau)$ using the linear theory from Section 3.4 to iteratively adjust approximate solutions. The proof of Theorem 1.1 will be complete once Proposition 3.4 has been established.

We first make a general remark about the quadratic remainder term in the Taylor expansion of $h \mapsto \Theta(g+h)$ at an arbitrary $C^{2,\alpha}(\bar{\Omega})$ metric g . We have that

$$\Theta(g+h) = (R(g+h), V(g+h)) = (R(g), V(g)) + \mathcal{S}_g(h) + Q_g(h)$$

where $\mathcal{S}_g = D\Theta_g$ is the linearization of Θ at g and where Q_g is the “quadratic remainder” term. More precisely, in a fixed coordinate system, $\mathcal{S}_g(h)$ (respectively $Q_g(h)$) is a homogeneous linear (quadratic) polynomial in $h_{ij}, \partial_k h_{ij}$ and $\partial_{kl}^2 h_{ij}$ whose coefficients are smooth functions of $g_{ij}, \partial_k g_{ij}, \partial_{kl}^2 g_{ij}$ (and $h_{ij}, \partial_k h_{ij}, \partial_{kl}^2 h_{ij}$). It follows that there is a constant $D > 0$ so that for any open subset $U \subset \subset \Omega$ we have that $\|Q_g(h)\|_{C^\alpha(U) \times \mathbb{R}} \leq D\|h\|_{C^{2,\alpha}(U)}^2$. Using this estimate for $U = B(x, \phi(x))$ a small ball near the boundary, and for U the complement of a thin collar neighborhood of $\partial\Omega$, we obtain

$$\|Q_g(h)\|_0 \leq D\|h\|_2^2$$

where D might have changed. Here, we also used that the weight ρ tends to zero faster on approach to the boundary than any power of the distance function. Enlarging D slightly if necessary, we also see that

$$(3.7) \quad \|Q_\gamma(h)\|_0 \leq D\|h\|_2^2$$

holds for every metric γ that is sufficiently close to g in $C^{2,\alpha}(\bar{\Omega})$. Similarly, we have that

$$(3.8) \quad \|\mathcal{S}_\gamma(h) - \mathcal{S}_{\gamma'}(h)\|_0 \leq D\|h\|_2 \|\gamma - \gamma'\|_2$$

provided that γ, γ' are $C^{2,\alpha}(\overline{\Omega})$ close to g . In (3.7) and (3.8) the weighted L^2 and Schauder norms, in whose definition we use the distance function to the boundary of Ω , are computed with respect to the fixed metric g , cf. Remark 3.6.

Proposition 3.4. *Let $(\overline{\Omega}, g)$ be as in Theorem 1.1. Let $C > 0$ be the constant from Proposition 3.3. There exists $\epsilon_0 > 0$ so that given any $(\sigma, \tau) \in \mathcal{B}_0$ with $\|(\sigma, \tau)\|_0 \leq \epsilon_0$, there exists $(u, a) \in \mathcal{B}_4$ so that for $h = \rho \mathcal{S}_g^*(u, a)$, $g + h$ is a metric with $\Theta(g + h) = \Theta(g) + (\sigma, \tau)$, and such that $\|(u, a)\|_4 \leq 2C\|(\sigma, \tau)\|_0$ and $\|h\|_2 \leq 2C\|(\sigma, \tau)\|_0$.*

Proof. Let $(u_0, a_0) \in \mathcal{B}_4$ be the solution of $\mathcal{S}_g \rho \mathcal{S}_g^*(u_0, a_0) = (\sigma, \tau)$ from Proposition 3.3 and let $h_0 = \rho \mathcal{S}_g^*(u_0, a_0)$, so that

$$\|(u_0, a_0)\|_4 \leq C\|(\sigma, \tau)\|_0 \text{ and } \|h_0\|_2 \leq C\|(\sigma, \tau)\|_0.$$

From (3.7) we obtain that $\|Q_g(h_0)\|_0 \leq D\|h_0\|_2^2$ and hence

$$\|\Theta(g + h_0) - (R(g) + \sigma, V(g) + \tau)\|_0 = \|Q_g(h_0)\|_0 \leq DC^2\|(\sigma, \tau)\|_0^2.$$

We let $\gamma_1 := g + h_0$. Note that γ_1 is a $C^{2,\alpha}(\overline{\Omega})$ metric provided $\|(\sigma, \tau)\|_0$ is sufficiently small. Fix $\delta \in (0, 1)$. We require that $\epsilon_0 > 0$ to be so small that $DC^2\epsilon_0^{1-\delta} \leq 1$. We now proceed inductively:

Lemma 3.5 (Cf. [7, Proposition 3.9]). *Fix $\delta \in (0, 1)$. Let C be the constant from Proposition 3.3. There exists $\epsilon_0 \in (0, \frac{1}{2})$ depending only on δ, Ω , and $g \in C^{4,\alpha}(\overline{\Omega})$ such that the following holds. Suppose that $m \geq 1$ and that we have constructed $(u_0, a_0), \dots, (u_{m-1}, a_{m-1}) \in \mathcal{B}_4$, $h_0, \dots, h_{m-1} \in \mathcal{B}_2$ where $h_p = \rho \mathcal{S}_g^*(u_p, a_p)$, and metrics $\gamma_1, \dots, \gamma_m \in C^{2,\alpha}(\overline{\Omega})$ where $\gamma_j = g + \sum_{p=0}^{j-1} h_p$. Assume that $\|(\sigma, \tau)\|_0 \leq \epsilon_0$ and that for all $0 \leq p \leq m-1$,*

$$(3.9) \quad \|(u_p, a_p)\|_4 \leq C\|(\sigma, \tau)\|_0^{(1+p\delta)} \quad \text{and} \quad \|h_p\|_2 \leq C\|(\sigma, \tau)\|_0^{(1+p\delta)},$$

and that for all $1 \leq j \leq m$,

$$(3.10) \quad \|\Theta(\gamma_j) - (R(g) + \sigma, V(g) + \tau)\|_0 \leq \|(\sigma, \tau)\|_0^{(1+j\delta)}.$$

If we define $h_m := \rho \mathcal{S}_g^(u_m, a_m)$ where (u_m, a_m) is the variational solution to $\mathcal{S}_g \rho \mathcal{S}_g^*(u_m, a_m) = (R(g) + \sigma, V(g) + \tau) - \Theta(\gamma_m)$ from Proposition 3.3, and if we let $\gamma_{m+1} := \gamma_m + h_m$, then γ_{m+1} is a $C^{2,\alpha}(\overline{\Omega})$ metric and the estimates (3.9) and (3.10) hold for $p = m$ and $j = m+1$.*

Proof. We let $\gamma_0 := g$. The induction hypotheses ensure that $\|g - \gamma_j\|_{C^{2,\alpha}(\overline{\Omega})}$ stays small (depending on $\epsilon_0 > 0$) throughout the iteration. Using Proposition 3.3, we find $(u_m, a_m) \in \mathcal{B}_4$ such that $\mathcal{S}_g \rho \mathcal{S}_g^*(u_m, a_m) = (R(g) + \sigma, V(g) + \tau) - \Theta(\gamma_m)$. Putting $h_m := \rho \mathcal{S}_g^*(u_m, a_m)$, the hypotheses imply the following:

$$\begin{aligned} \|(u_m, a_m)\|_4 &\leq C\|(R(g) + \sigma, V(g) + \tau) - \Theta(\gamma_m)\|_0 \leq C\|(\sigma, \tau)\|_0^{1+m\delta}, \\ \|h_m\|_2 &\leq C\|(R(g) + \sigma, V(g) + \tau) - \Theta(\gamma_m)\|_0 \leq C\|(\sigma, \tau)\|_0^{1+m\delta}. \end{aligned}$$

Note that

$$\begin{aligned} \Theta(\gamma_{m+1}) &= \Theta(\gamma_m) + \mathcal{S}_{\gamma_m}(h_m) + Q_{\gamma_m}(h_m) \\ &= (R(g) + \sigma, V(g) + \tau) + \sum_{p=0}^{m-1} [\mathcal{S}_{\gamma_{p+1}}(h_m) - \mathcal{S}_{\gamma_p}(h_m)] + Q_{\gamma_m}(h_m). \end{aligned}$$

Using (3.7), (3.8) and elementary manipulations, we obtain that

$$\begin{aligned}
\|(R(g) + \sigma, V(g) + \tau) - \Theta(\gamma_{m+1})\|_0 &\leq D \left(\|h_m\|_2^2 + \|h_m\|_2 \sum_{p=0}^{m-1} \|h_p\|_2 \right) \\
&\leq DC^2 \left(\|(\sigma, \tau)\|_0^{(2+2m\delta)} + \|(\sigma, \tau)\|_0^{2+m\delta} \sum_{p=0}^{m-1} \|(\sigma, \tau)\|_0^{\delta p} \right) \\
&\leq 2DC^2 \epsilon_0^{1-\delta} (1 - \epsilon_0^\delta)^{-1} \|(\sigma, \tau)\|_0^{1+(m+1)\delta}.
\end{aligned}$$

Choose $\epsilon_0 > 0$ small enough so that $2DC^2 \epsilon_0^{1-\delta} (1 - \epsilon_0^\delta)^{-1} \leq 1$. \square

It follows that the series $\sum_{p=0}^{\infty} (u_p, a_p)$ converges in \mathcal{B}_4 to some (u, a) , and that if $h := \rho \mathcal{S}_g^*(u, a)$, then $\gamma := g + h$ satisfies $\Theta(\gamma) = (R(g) + \sigma, V(g) + \tau)$. Choosing $\epsilon_0 > 0$ even smaller if necessary, we obtain that $\|(u, a)\|_4 \leq 2C\|(\sigma, \tau)\|_0$ and $\|h\|_2 \leq 2C\|(\sigma, \tau)\|$ from summing the estimates for (u_p, a_p) and h_p . This concludes the proof of Proposition 3.4. \square

Remark 3.6. The conclusion of Proposition 3.4 holds with one choice for $\epsilon_0 > 0$ and $C > 0$ for any metric g' from a small $C^{4,\alpha}(\bar{\Omega})$ neighborhood of g . To see this, note that the condition that $\mathcal{S}_{g'}$ have only trivial kernel in $H_{\text{loc}}^1(\Omega) \times \mathbb{R}$ is an open condition for $g' \in C^{4,\alpha}(\bar{\Omega})$. This follows easily from Proposition 2.2. The fundamental coercivity estimate (3.3), and hence (3.1), holds with a uniform constant C for all metrics g' that are close to g in $C^{4,\alpha}(\bar{\Omega})$. The dependence on the metric can easily be made part of the proof. The derivation of (3.3) is the only indirect argument that was used in the proof of Proposition 3.4. We emphasize that the norms of the lower order terms of the operators to which we apply Schauder estimates in the proof of Proposition 3.3 are uniformly bounded in appropriate spaces, even though the weighted norms as g' varies in a neighborhood of g are not necessarily equivalent. Thus there is a constant C for which the weighted Schauder estimates will hold for all g' from a $C^{4,\alpha}(\bar{\Omega})$ neighborhood of g .

3.6. Continuous dependence.

Proposition 3.7. *There exist $\epsilon_0 > 0$ and $C > 0$ with the following property. If $(\sigma_i, \tau_i) \in \mathcal{B}_0$ with $\|(\sigma_i, \tau_i)\|_0 \leq \epsilon_0$ for $i = 1, 2$, and if γ_1 and γ_2 are the corresponding solutions of $\Theta(\gamma_i) = \Theta(g) + (\sigma_i, \tau_i)$, $i = 1, 2$, constructed in the proof of Proposition 3.4, then $\|\gamma_1 - \gamma_2\|_2 \leq C\|(\sigma_1, \tau_1) - (\sigma_2, \tau_2)\|_0$.*

Proof. Let (u_i, a_i) , $h_i = \rho \mathcal{S}_g^*(u_i, a_i)$, and $\gamma_i = g + h_i$ be as in Proposition 3.4. Then

$$\mathcal{S}_g(h_1 - h_2) = \mathcal{S}_g \rho \mathcal{S}_g^*(u_1 - u_2, a_1 - a_2) = (\sigma_1 - \sigma_2, \tau_1 - \tau_2) + (Q_g(h_2) - Q_g(h_1)).$$

Analysis of the remainder term in the Taylor expansion as in Section 3.5 gives

$$\|Q_g(h_1) - Q_g(h_2)\|_0 \leq D\|h_1 - h_2\|_2(\|h_1\|_2 + \|h_2\|_2) \leq 2CD\epsilon_0\|h_1 - h_2\|_2.$$

Interior Schauder estimates for the operator $\rho^{-1}L_g\rho L_g^*$ give that

$$\begin{aligned}
\|h_1 - h_2\|_2 &= \|\rho \mathcal{S}_g^*(u_1 - u_2, a_1 - a_2)\|_2 \leq C\|(u_1 - u_2, a_1 - a_2)\|_4 \\
&\leq C(\|\rho^{-1}(\sigma_1 - \sigma_2) - \rho^{-1}\frac{a_1 - a_2}{2}L_g(\rho g)\|_{C_{\phi, \phi^4 + \frac{5}{2}\rho^{1/2}}^{0,\alpha}} \\
&\quad + \|(u_1 - u_2, a_1 - a_2)\|_{H_\rho^2 \times \mathbb{R}} + \|Q_g(h_1) - Q_g(h_2)\|_0) \\
(3.11) \quad &\leq C(\|\sigma_1 - \sigma_2\|_0 + \|(u_1 - u_2, a_1 - a_2)\|_{H_\rho^2 \times \mathbb{R}} + 2CD\epsilon_0\|h_1 - h_2\|_2).
\end{aligned}$$

By the coercivity estimate (3.1),

$$\begin{aligned} \|(u_1 - u_2, a_1 - a_2)\|_{H_\rho^2 \times \mathbb{R}}^2 &\leq C \int_{\Omega} \mathcal{S}_g^*(u_1 - u_2, a_1 - a_2) \cdot \rho \mathcal{S}_g^*(u_1 - u_2, a_1 - a_2) d\mu_g \\ &= C \int_{\Omega} \mathcal{S}_g^*(u_1 - u_2, a_1 - a_2) \cdot (h_1 - h_2) d\mu_g. \end{aligned}$$

We would like to integrate by parts in the last term. Since $\mathcal{S}_g^*(u_1 - u_2, a_1 - a_2) \in L_\rho^2(\Omega) \cap C_{\phi, \phi^{2+\frac{n}{2}}\rho^{\frac{1}{2}}}^{2,\alpha}(\Omega)$, it is not immediately clear that the boundary terms will vanish. Using that $C^\infty(\overline{\Omega})$ is dense in $H_\rho^2(\Omega)$ (cf. [9, Lemma 2.1]), we can justify the integration by parts using an approximation argument. It follows that

$$\begin{aligned} \|(u_1 - u_2, a_1 - a_2)\|_{H_\rho^2 \times \mathbb{R}}^2 &\leq C \int_{\Omega} (u_1 - u_2, a_1 - a_2) \cdot \mathcal{S}_g(h_1 - h_2) d\mu_g \\ &\leq C \|(u_1 - u_2, a_1 - a_2)\|_{L_\rho^2 \times \mathbb{R}} \|(\sigma_1 - \sigma_2, \tau_1 - \tau_2) + (Q_g(h_2) - Q_g(h_1))\|_{L_{\rho^{-1}}^2 \times \mathbb{R}} \\ &\leq C \|(u_1 - u_2, a_1 - a_2)\|_{H_\rho^2 \times \mathbb{R}} (\|(\sigma_1 - \sigma_2, \tau_1 - \tau_2)\|_0 + 2CD\epsilon_0 \|h_1 - h_2\|_2). \end{aligned}$$

Together with (3.11), this completes the proof. \square

3.7. Higher order regularity of the solution and the proof of Theorem 1.2.

The non-linear differential operator $u \mapsto \hat{P}(u) = \rho^{-1} (R(g + \rho \mathcal{S}_g^*(u, a)) - R(g))$ is quasi-linear fourth order elliptic in u provided that $\rho \mathcal{S}_g^*(u, a)$ is sufficiently small. The fourth order part of this operator is equal to

$$\frac{1}{2} \gamma^{i\ell} \gamma^{jk} g^{ab} (-g_{j\ell} u_{abik} + g_{jk} u_{abil} + g_{i\ell} u_{abjk} - g_{ik} u_{abj\ell}).$$

Here, $\gamma^{ij} := (g + \rho \mathcal{S}_g^*(u, a))_{ij}^{-1}$. To see ellipticity, note that for $\rho \mathcal{S}_g^*(u, a)$ sufficiently small, γ^{ij} is close to g^{ij} and the symbol of the operator is close to $(n-1)|\xi|^4$. The lower order terms may blow up on the boundary. The equation $\hat{P}(u) = \rho^{-1}\sigma$ can be cast in a form to which higher order Schauder estimates similar to those in (A.2) of Appendix A and bootstrapping can be applied. Note that the right hand side is compactly supported and hence lies in any of the weighted Sobolev spaces we defined. Under the regularity assumptions for $(\overline{\Omega}, g)$ in the statement of Theorem 1.2, we obtain that

$$(3.12) \quad \|u\|_{C_{\phi, \phi^{n/2}\rho^{1/2}}^{k+4, \alpha}} \leq C(k, \Omega, g) \left(\|\hat{P}u\|_{C_{\phi, \phi^{4+\frac{n}{2}}\rho^{1/2}}^{k, \alpha}} + \|u\|_{L_\rho^2} \right).$$

This implies that

$$\|h\|_{C_{\phi, \phi^{2+\frac{n}{2}}\rho^{-1/2}}^{k+2, \alpha}} = \|\rho \mathcal{S}_g^*(u, a)\|_{C_{\phi, \phi^{2+\frac{n}{2}}\rho^{-1/2}}^{k+2, \alpha}} \leq C\|(u, a)\|_{C_{\phi, \phi^{n/2}\rho^{1/2}}^{k+4, \alpha} \times \mathbb{R}} < \infty.$$

In particular, it follows that $h = \rho \mathcal{S}_g^*(u, a)$ extends by 0 as a $C^{k+2, \alpha}$ function across the boundary of Ω . We emphasize that we lose two degrees of differentiability in the construction of h . Theorem 1.2 follows from this, Remark 3.6, and inspection of how the weighted norms we have used are constructed and depend on the metric tensor. Note that to arrange $\text{supp}(\gamma - g)$ to be compactly contained in Ω , we would first replace Ω by Ω_δ , for $\delta > 0$ so small that $\Omega_0 \subset \Omega_\delta$ and so that Ω_δ is not V -static (see the proof of Proposition 3.1).

We remark that if the metric g is smooth to start with, then we can bootstrap to conclude that h , and hence $\gamma = g + h$, is also smooth.

4. CONSTANT SCALAR CURVATURE GLUING WITH A VOLUME CONSTRAINT

Theorem 4.1. *Let $k \geq 2$, $n \geq 3$, and $\sigma_n \in \{-n(n-1), 0, n(n-1)\}$. Let $(\Sigma_1, \gamma_1), (\Sigma_2, \gamma_2)$ be two compact n -dimensional $C^{k,\alpha}$ Riemannian manifolds with non-empty boundary. Assume that $R(\gamma_1) = R(\gamma_2) = \sigma_n$. When $\sigma_n > 0$, we also assume that the operators $(\Delta_{\gamma_i} + n)$ have positive Dirichlet spectrum on Σ_i . Let $p_i \in \text{int}(\Sigma_i)$ and U_i be a neighborhood of p_i in Σ_i for $i \in \{1, 2\}$. There is a family of $C^{k,\alpha}$ metrics $\{\hat{\gamma}_T\}$ on the connected sum $\Sigma_1 \# \Sigma_2 \supset (\Sigma_1 \setminus U_1) \sqcup (\Sigma_2 \setminus U_2)$ with $R(\hat{\gamma}_T) = \sigma_n$ and such that $\hat{\gamma}_T \rightarrow \gamma_1 \sqcup \gamma_2$ in $C^{k,\alpha}((\Sigma_1 \setminus U_1) \sqcup (\Sigma_2 \setminus U_2))$ and $\text{vol}(\Sigma_1 \# \Sigma_2, \hat{\gamma}_T) \rightarrow \text{vol}(\Sigma_1, \gamma_1) + \text{vol}(\Sigma_2, \gamma_2)$ as $T \rightarrow \infty$.*

Remark 4.2. It is clear from the proof that Theorem 4.1 also holds in the case where $\partial\Sigma_i = \emptyset$ and $\sigma_n < 0$ and in the case where $\sigma_n > 0$ and $(\Delta_{\gamma_i} + n)$ has positive spectrum.

Theorem 4.1 is a augments the result of [5, Theorem 1.2], where an analogous gluing result is formulated for $\sigma_n \leq 0$, without a volume constraint. We include here the case $\sigma_n > 0$, and we also estimate the volume, which we need for our application to Theorem 1.6. The proof of Theorem 4.1 follows the approach of the proofs of [13, Theorem 1] and [14, Theorem 3.10] closely. For completeness and clarity, we repeat many of the arguments rather than simply indicating necessary modifications. Some of our estimates are slightly sharper than the analogues in [13, 14], and there is at least one technical simplification as we do not need to employ weighted Hölder spaces in our argument.

4.1. The approximate solution. Here we construct approximate solutions to the scalar curvature equation on the connected sum. To do this, we use a conformal rescaling in each of two punctured geodesic balls $B_i \setminus \{p_i\} \subset U_i$ to produce a metric on each of $\Sigma_i \setminus \{p_i\}$ with an asymptotically cylindrical end. We identify these ends (after a cut off to an exact cylindrical metric far along the end) to form the connected sum. We then superimpose the two conformal factors used to produce these cylindrical blow ups on the ends to obtain a new conformal factor. This gluing generalizes how the (scalar flat) Schwarzschild metric $(\mathbb{R}^n \setminus \{0\}, (1 + \frac{m}{2|x|^{n-2}})^{\frac{4}{n-2}} \sum_{j=1}^n dx_j^2)$ connects two copies of Euclidean space through a small neck of cross sectional area proportional to $m^{(n-1)/(n-2)}$. Our construction of approximate solutions here follows that of [13, Section 2] very closely.

Lemma 4.3 (Quasi-polar and quasi-cylindrical coordinates). *Let $n, k \geq 2$, let (M, g) be an n -dimensional $C^{k,\alpha}$ Riemannian manifold, and let $p \in \text{int}(M)$. There exists $r_0 > 0$ such that for every $\rho \in (0, r_0)$, there exist $C^{k+1,\alpha}$ coordinates (x^1, \dots, x^n) on an open subset containing $B(p, \rho/2)$, with $(0, \dots, 0)$ corresponding to p , and such that $g_{ij} = \delta_{ij} + Q_{ij}$, where $Q_{ij} \in C^{k,\alpha}(B(p, \rho/2))$, with $Q_{ij}(0) = 0 = Q_{ij,\ell}(0)$ for all $i, j, \ell \in \{1, \dots, n\}$. Let (r, θ) be polar coordinates in this coordinate system. One can extend r to all of $M \setminus \{p\}$ as a $C^{k,\alpha}$ function with uniform bounds on $r^{\ell-1} |\nabla_g^\ell r|_g$ for $\ell = 1, \dots, k$, so $r(x)$ agrees with $\text{dist}_g(p, x)$ on $B(p, \rho) \setminus B(p, \rho/2)$, such that $\frac{1}{2}r(x) \leq \text{dist}_g(p, x) \leq 2r(x)$ on all of $B(p, \rho)$, and so that $\frac{r(x)}{\text{dist}_g(p, x)} \rightarrow 1$*

as $\text{dist}_g(p, x) \rightarrow 0$. Changing to cylindrical coordinates $t(x) = -\log r(x)$, we can express the metric $r^{-2}g$ on $B(p, \rho/2) \setminus \{p\}$ in the form $dt^2 + g_{\mathbb{S}^{n-1}} + e^{-2t} \hat{h}$ where $\hat{h} \in C_{\text{loc}}^{k, \alpha}(B(p, \rho/2) \setminus \{p\})$. Moreover, $\|\hat{h}\|_{C^{k, \alpha}([-\log(\rho/2), \infty) \times \mathbb{S}^{n-1})} < \infty$, where the norm (including covariant derivatives) is taken with respect to the cylindrical metric $dt^2 + g_{\mathbb{S}^{n-1}}$.

Proof. We can compose any $C^{k+1, \alpha}$ diffeomorphism φ of a neighborhood of p in M onto a neighborhood of the origin $\varphi(p)$ in \mathbb{R}^n with a map that is a non-singular linear transformation plus a vector field whose entries are homogenous quadratic polynomials in the coordinates, to obtain a new coordinate system centered at p in which $g_{ij} = \delta_{ij} + Q_{ij}$, with $Q_{ij}(0) = 0 = Q_{ij, \ell}(0)$ for all $i, j, \ell \in \{1, \dots, n\}$. Let $\theta : \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n$ be the standard embedding of the unit sphere, and let $\Phi : \mathbb{R} \times \mathbb{S}^{n-1} \rightarrow \mathbb{R}^n \setminus \{(0, \dots, 0)\}$ be the ‘‘cylindrical coordinates map’’ $\Phi(t, \theta) = e^{-t}\theta$. Pulling back g_{ij} by this map, we get

$$\begin{aligned} \Phi^*(g_{ij} dx^i \otimes dx^j) &= e^{-2t} (dt^2 + g_{\mathbb{S}^{n-1}} + Q_{ij}(e^{-t}\theta)(-\theta^i dt + d\theta^i) \otimes (-\theta^j dt + d\theta^j)) \\ &=: e^{-2t} (dt^2 + g_{\mathbb{S}^{n-1}} + e^{-2t} \hat{h}). \end{aligned}$$

Using $Q_{ij}(0) = 0 = Q_{ij, \ell}(0)$, the assertions about the decay of \hat{h} follow readily. \square

Remark 4.4. In [13, (9)], the weaker decay rate $e^{-2t}(dt^2 + g_{\mathbb{S}^{n-1}} + e^{-t}\hat{h})$ with \hat{h} and its derivatives bounded as $t \rightarrow \infty$ is used. Our sharper estimate leads to better bounds in some places than those obtained in [13].

Remark 4.5. We do not work with cylindrical coordinates based on geodesic polar coordinates in Lemma 4.3 to avoid an unnecessary loss of regularity. Recall that the distance function of a $C^{k, \alpha}$ metric to a point p is $C^{k, \alpha}$ in a punctured neighborhood of p , see [11].

We now fix $R \in (0, \frac{1}{3} \min\{1, \text{dist}_{\gamma_i}(p_i, \partial \Sigma_i), (r_0)_i, i = 1, 2\})$. Here, $(r_0)_i$ is as in Lemma 4.3 applied with $M = \Sigma_i$ and $p = p_i$. We let $r_{(i)}(x)$ be the functions constructed in Lemma 4.3 for $\rho = R$.

We define $r_i(x) = \min\{2R, r_{(i)}(x)\}$. Let $\psi : (0, \infty) \rightarrow (0, \infty)$ be a smooth, positive function with

$$\psi(t) = \begin{cases} t^{\frac{n-2}{2}} & 0 < t < R \\ \text{interpolation} & R \leq t \leq 2R \\ 1 & t > 2R. \end{cases}$$

We let $\Psi_i(x) = \psi(r_i(x))$. On $B_{\gamma_i}(p_i, R)$, $(\Psi_i(x))^{\frac{4}{n-2}} = (r_i(x))^2$. Let $\Sigma_i^* = \Sigma_i \setminus \{p_i\}$. Then $(\Sigma_i^*, \Psi_i^{-\frac{4}{n-2}} \gamma_i)$ is $(\Sigma_i \setminus B_{\gamma_i}(p_i, 2R), \gamma_i)$ with an infinite, asymptotically cylindrical end attached.

Let $T \geq T_0 \gg -2 \log R > 1$, and let (r_i, θ) be the quasi-geodesic polar coordinates on $B_{\gamma_i}(p_i, 3R) \subset \Sigma_i$ of Lemma 4.3. Let $s_i = -\log r_i + \log R - \frac{T}{2}$. Note that under this change of variables, $r_i = R$ corresponds to $s_i = -\frac{T}{2}$ and $r_i \searrow 0$ corresponds to $s_i \nearrow \infty$. By Lemma 4.3, the metric $\Psi_i^{-4/(n-2)} \gamma_i$ on $B_{\gamma_i}(p_i, R)$ can be written as $ds_i^2 + g_{\mathbb{S}^{n-1}} + e^{-T} e^{-2s_i} R^2 \hat{h}_i$ where \hat{h}_i and its covariant derivatives with respect to the cylindrical metric $ds_i^2 + g_{\mathbb{S}^{n-1}}$ are bounded, independently of T . Let $\gamma_{i, T}$ be the metric obtained by transitioning smoothly in the (s_i, θ) -region $(-1, -\frac{1}{2}) \times \mathbb{S}^{n-1}$ from the metric $\Psi_i^{-4/(n-2)} \gamma_i = ds_i^2 + g_{\mathbb{S}^{n-1}} + e^{-T} e^{-2s_i} R^2 \hat{h}_i$ on

$(-\frac{T}{2}, -1) \times \mathbb{S}^{n-1}$ to the exact cylindrical metric $ds_i^2 + g_{\mathbb{S}^{n-1}}$ on $(-\frac{1}{2}, \infty) \times \mathbb{S}^{n-1}$. Such a transition can be accomplished using a cut-off function whose norms do not depend on T .

The cylindrical ends of the two Riemannian manifolds so obtained can be identified by forming the quotient via the relation $(s_1, \theta) \sim (-s_2, \theta)$ in the exactly cylindrical pieces $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{S}^{n-1}$ in each of $(\Sigma_i^*, \gamma_{i,T})$. We obtain a new manifold (Σ_T, γ_T) , where Σ_T is topologically just $\Sigma_1 \# \Sigma_2$. We define a new linear coordinate s on $[-\frac{T}{2}, \frac{T}{2}] \times \mathbb{S}^{n-1} \cong C_T \subset \Sigma_T$, so that $s = s_1$ for $s \leq 0$ and $s = -s_2$ for $s \geq 0$. On C_T , the metric γ_T takes the form

$$(4.1) \quad \gamma_T = ds^2 + g_{\mathbb{S}^{n-1}} + e^{-T} \cosh(2s) \hat{h}_T,$$

where \hat{h}_T and its derivatives with respect to the cylindrical metric are bounded independently of T , and where $\hat{h}_T = 0$ on $[-\frac{1}{2}, \frac{1}{2}] \times \mathbb{S}^{n-1}$.

Let $\chi_{1,T}(s)$ be a cut-off function transitioning smoothly from 1 near $\{\frac{T}{2} - 1\} \times \mathbb{S}^{n-1}$ to 0 near $\{\frac{T}{2}\} \times \mathbb{S}^{n-1}$, and let $\chi_{2,T}(s)$ be the corresponding cut-off function transitioning smoothly from 1 near $\{-\frac{T}{2} + 1\} \times \mathbb{S}^{n-1}$ to 0 near $\{-\frac{T}{2}\} \times \mathbb{S}^{n-1}$. The function Ψ_T defined by $\Psi_T(s, \theta) := \chi_{1,T}(s)\psi(Re^{-s-\frac{T}{2}}) + \chi_{2,T}(s)\psi(Re^{s-\frac{T}{2}})$ extends smoothly to Σ_T .

Note that on $(-\frac{T}{2} + 1, \frac{T}{2} - 1) \times \mathbb{S}^{n-1} \subset C_T$, we have

$$\Psi_T(s, \theta) = (Re^{-s-\frac{T}{2}})^{\frac{n-2}{2}} + (Re^{s-\frac{T}{2}})^{\frac{n-2}{2}} = 2R^{\frac{n-2}{2}} e^{-\frac{(n-2)T}{4}} \cosh\left(\frac{(n-2)s}{2}\right).$$

The scalar curvature of $\Psi_T^{4/(n-2)}\gamma_T$ is given by

$$(4.2) \quad R(\Psi_T^{\frac{4}{n-2}}\gamma_T) = c_n^{-1}\Psi_T^{-\frac{n+2}{n-2}}(-\Delta_{\gamma_T}\Psi_T + c_n R(\gamma_T)\Psi_T).$$

Here, $c_n = \frac{n-2}{4(n-1)}$. Because γ_T is exactly cylindrical on $(-\frac{1}{2}, \frac{1}{2}) \times \mathbb{S}^{n-1}$, we have that $R(\Psi_T^{4/(n-2)}\gamma_T) = 0$ there. In fact, on this region, $\Psi_T^{4/(n-2)}\gamma_T$ is precisely the *Schwarzschild metric* of mass $m = 2R^{(n-2)}e^{-(n-2)T/2}$, with the minimal sphere at $s = 0$ in our coordinates (cf. [2]). The geometry of $(\Sigma_T, \Psi_T^{4/(n-2)}\gamma_T)$ is thus that of a Schwarzschild neck together with interpolating regions joining $(\Sigma_1 \setminus B_{\gamma_1}(p_1, 2R), \gamma_1)$ and $(\Sigma_2 \setminus B_{\gamma_2}(p_2, 2R), \gamma_2)$.

We use $\{(\Sigma_T, \Psi_T^{4/(n-2)}\gamma_T)\}_{T \geq T_0}$ as a family of approximate solutions to the constant scalar curvature equation on $\Sigma_1 \# \Sigma_2$ that we want to perturb to obtain exact solutions. The injectivity radius of (Σ_T, γ_T) is bounded below uniformly as $T \rightarrow \infty$. Define the operator

$$\mathcal{N}_T(f) := -\Delta_{\gamma_T}f + c_n R(\gamma_T)f - c_n \sigma_n f^{\frac{n+2}{n-2}}.$$

In view of (4.2), we would like to solve $\mathcal{N}_T(\Psi_T + \eta_T) = 0$ for small η_T , such that $\Psi_T + \eta_T > 0$. To accomplish this by perturbation, we will estimate $\mathcal{N}_T(\Psi_T)$ and analyze the mapping properties of the linearization \mathcal{L}_T of \mathcal{N}_T at Ψ_T .

Let $\|f\|_{k,\alpha} := \|f\|_{C^{k,\alpha}(\Sigma_T)}$ denote the Hölder norm on (Σ_T, γ_T) . Let “ \lesssim ” denote an inequality up to multiplication by a constant that is independent of T . We begin by estimating $\mathcal{N}_T(\Psi_T)$:

Proposition 4.6 (Cf. [13, Proposition 6]). *If $2 \leq n \leq 6$, we have that*

$$(4.3) \quad \|\mathcal{N}_T(\Psi_T)\|_{k-2,\alpha} \lesssim e^{-\frac{(n-2)T}{2}}.$$

For $n > 6$,

$$(4.4) \quad \|\mathcal{N}_T(\Psi_T)\|_{k-2,\alpha} \lesssim e^{-\frac{(n+2)T}{4}}.$$

This estimate follows along the lines of [13, Proposition 6], [14, Proposition 3.6]. For the reader's convenience, we include a proof in Appendix B. The cited proofs involve some additional terms coming from the Einstein constraint equations. Our estimates are also slightly sharper due in part to the fact that we use a better estimate on the metric (Lemma 4.3).

4.2. The linearized operator. The linearization \mathcal{L}_T of \mathcal{N}_T at Ψ_T is given by

$$\mathcal{L}_T(f) := D\mathcal{N}_T|_{\Psi_T}(f) = -\Delta_{\gamma_T} f + c_n R(\gamma_T) f - c_n \sigma_n \frac{n+2}{n-2} \Psi_T^{4/(n-2)} f.$$

For a given integer ℓ with $0 \leq \ell \leq k$, we define the function space $\mathring{C}^{\ell,\alpha}(\Sigma_T) = \{u \in C^{\ell,\alpha}(\Sigma_T) \text{ and } u|_{\partial\Sigma_T} = 0\}$. The proof of the following proposition is very similar to [13, 14]. We include the proof for completeness and clarity. We let $\Sigma_{i,r}^* = \Sigma_i \setminus B_{\gamma_i}(p_i, r)$.

Proposition 4.7 (Cf. [13, Proposition 8]). *For every $k \geq 2$ and all sufficiently large T , the operators $\mathcal{L}_T : \mathring{C}^{k,\alpha}(\Sigma_T) \rightarrow C^{k-2,\alpha}(\Sigma_T)$ are invertible. The norms of the inverse operators $\mathcal{G}_T : C^{k-2,\alpha}(\Sigma_T) \rightarrow \mathring{C}^{k,\alpha}(\Sigma_T)$ are uniformly bounded.*

Proof. We first show that $\mathcal{L}_T : \mathring{C}^{2,\alpha}(\Sigma_T) \rightarrow C^{0,\alpha}(\Sigma_T)$ is invertible for large T . The invertibility of these operators for $k \geq 3$ follows from elliptic regularity. This map is Fredholm of index zero, so we only have to show that it is injective for T large enough. We do this by contradiction below. First, note that $s = (-1)^j$ corresponds to $r_j = Re^{1-\frac{T}{2}} =: r(T)$, and that $(\Sigma_{j,r(T)}^*, \Psi_j^{-4/(n-2)} \gamma_j) \subset (\Sigma_T, \gamma_T)$. Thus, as T grows, more and more of $(\Sigma_j^*, \Psi_j^{-4/(n-2)} \gamma_j)$ is contained in (Σ_T, γ_T) .

Suppose there is a sequence $T_m \nearrow \infty$ and non-zero $\eta_m \in \mathring{C}^{2,\alpha}(\Sigma_{T_m})$ so that $\mathcal{L}_{T_m}(\eta_m) = 0$. By normalization, we may arrange $\max_{\Sigma_{T_m}} |\eta_m| = 1$. We distinguish two cases.

In the first case, we assume that for one of $j = 1, 2$, and for some $0 < r < R$, there is a $c > 0$ so that $\max_{\Sigma_{j,r}^*} |\eta_m| \geq c$ (at least for a subsequence, which we re-index).

Let $\tilde{\gamma}_j := \Psi_j^{-4/(n-2)} \gamma_j$. The operators \mathcal{L}_{T_m} converge locally on Σ_j^* to the operator $\mathcal{L}_j = -\Delta_{\tilde{\gamma}_j} + c_n R(\tilde{\gamma}_j) - c_n \sigma_n \frac{n+2}{n-2} \Psi_j^{4/(n-2)}$. Since the η_m are uniformly bounded, interior Schauder estimates on the equations $\mathcal{L}_{T_m}(\eta_m) = 0$ imply that we can take a subsequence converging in C^2 on compact subsets of Σ_j^* , to a non-trivial limit function η on Σ_j^* . Moreover, we have $\mathcal{L}_j(\eta) = 0$ on Σ_j^* . Applying the identity

$$-\Delta_{\tilde{\gamma}_j} f + c_n R(\tilde{\gamma}_j) f = \Psi_j^{\frac{n+2}{n-2}} (-\Delta_{\gamma_j} (\Psi_j^{-1} f) + c_n R(\gamma_j) (\Psi_j^{-1} f))$$

for the conformal Laplacian valid for every $f \in C^2(\Sigma_j)$ with $f = \eta$, we obtain that

$$c_n \sigma_n \frac{n+2}{n-2} \Psi_j^{\frac{4}{n-2}} \eta = \Psi_j^{\frac{n+2}{n-2}} (-\Delta_{\gamma_j} (\Psi_j^{-1} \eta) + c_n R(\gamma_j) (\Psi_j^{-1} \eta)).$$

Since $R(\gamma_j) = \sigma_n$, we conclude that

$$0 = \Delta_{\gamma_j} (\Psi_j^{-1} \eta) + \frac{\sigma_n}{n-1} (\Psi_j^{-1} \eta).$$

Note that $0 \neq \Psi_j^{-1}\eta \in C^{2,\alpha}(\Sigma_j^*)$ vanishes on $\partial\Sigma_j$. We have that $|\Psi_j^{-1}\eta| \lesssim r_j^{-\frac{n-2}{2}}$ near p_j since $|\eta| \leq 1$. Our assumption that the operator $\Delta_{\gamma_j} + \frac{\sigma_n}{n-1}$ has positive Dirichlet spectrum on Σ_j implies the existence of a positive (Dirichlet) Green's function with pole at p_j , where it grows like $r_j^{-(n-2)}$. A standard application of the maximum principle shows that $|\Psi_j^{-1}\eta|$ lies below *any* positive multiple of this Green's function. Hence $\Psi_j^{-1}\eta \equiv 0$, a contradiction.

If we are not in the first case, then $\eta_m \rightarrow 0$ locally uniformly on Σ_j^* . If q_m is such that $|\eta_m(q_m)| = 1$, then $q_m = (s(q_m), \theta(q_m)) \in C_{T_m} \cong [-\frac{T_m}{2}, \frac{T_m}{2}] \times \mathbb{S}^{n-1}$ and $\min\{\frac{T_m}{2} - s(q_m), s(q_m) + \frac{T_m}{2}\} \rightarrow +\infty$. Introducing a new linear variable $\tilde{s} = s - s(q_m)$ (where we are now identifying the cylindrical pieces $C_{T_m} \subset \Sigma_{T_m}$ by identifying the (s, θ) coordinate patches), we conclude that $\gamma_{T_m} \rightarrow \dot{\gamma} = d\tilde{s}^2 + g_{\mathbb{S}^{n-1}}$ and $\Psi_{T_m} \rightarrow 0$ locally smoothly on $\mathbb{R} \times \mathbb{S}^{n-1}$. It follows that $\mathcal{L}_{T_m} \rightarrow \dot{\mathcal{L}} := -\Delta_{\dot{\gamma}} + c_n R(\dot{\gamma}) = -\Delta_{\dot{\gamma}} + \frac{(n-2)^2}{4}$ locally smoothly on $\mathbb{R} \times \mathbb{S}^{n-1}$. Using interior Schauder estimates and the supremum bound on η_m , we get a subsequence converging in C^2 on compact subsets of the cylinder to a solution η of $\dot{\mathcal{L}}(\eta) = 0$. Moreover, $|\eta| \leq 1$ and $|\eta| = 1$ at some point with $\tilde{s} = 0$. This contradicts the maximum principle.

Finally, we show that the norm of the inverse $\mathcal{G}_T : C^{k-2,\alpha}(\Sigma_T) \rightarrow \dot{C}^{k,\alpha}(\Sigma_T)$ can be bounded independently of T large. The proof proceeds by contradiction, as above. Suppose there are $T_m \nearrow \infty$ and $\eta_m \in C^{k-2,\alpha}(\Sigma_{T_m})$ so that $\|\eta_m\|_{k-2,\alpha} \rightarrow 0$ while $\|\mathcal{G}_{T_m}(\eta_m)\|_{k,\alpha} = 1$. Let $v_m = \mathcal{G}_{T_m}(\eta_m)$, so that $\|\mathcal{L}_{T_m}(v_m)\|_{k-2,\alpha} = \|\eta_m\|_{k-2,\alpha} \rightarrow 0$. Since $\|v_m\|_{k,\alpha} = 1$, we see that v_m converges in C^k on compact subsets of $\Sigma^* := \Sigma_1^* \cup \Sigma_2^*$. Just as above, there are two possible cases. In the first case, for either $j = 1$ or $j = 2$, some subsequence of the v_m converges in the C^k on compact subsets of Σ_j^* to a non-trivial solution v of the equation $\mathcal{L}_j(v) = 0$ with $v|_{\partial\Sigma_j} = 0$, in which case $\Psi_j^{-1}v$ extends to a non-trivial element in the kernel of $(\Delta_{\gamma_j} + \frac{\sigma_n}{n-1})$ on Σ_j . This is a contradiction.

In the second case, v_m converges to zero in C^k on any compact subset of $\Sigma_1^* \cup \Sigma_2^*$. The operators \mathcal{L}_T are uniformly elliptic. Since $\|\mathcal{L}_{T_m}(v_m)\|_{k-2,\alpha} \rightarrow 0$ and $\|v_m\|_{k,\alpha} = 1$, interior Schauder estimates imply that $\|v_m\|_0$ cannot tend to zero. Thus there is a $c > 0$ so that

$$c \leq \max_{C_{T_m}} |v_m| \leq 1.$$

The same rescaling to a cylinder as above leads to a contradiction. \square

We have now established the linear theory. Before moving on to the non-linear estimates, we note that in the zero scalar curvature case $\sigma_n = 0$, the problem we want to solve is linear: $\mathcal{N}_T(f) = -\Delta_{\gamma_T}(f) + c_n R(\gamma_T)f = \mathcal{L}_T(f)$. In this case, Proposition 4.7 is enough to obtain a solution $\eta_T \in \dot{C}^{k,\alpha}(\Sigma_T)$ of $\mathcal{N}_T(\Psi_T + \eta_T) = 0$, for sufficiently large T . Such an η_T is given by $\eta_T = -\mathcal{G}_T(\mathcal{N}_T(\Psi_T))$ and satisfies $\|\eta_T\|_{k,\alpha} \lesssim \|\mathcal{N}_T(\Psi_T)\|_{k-2,\alpha}$. By Proposition 4.6, we have that

$$(4.5) \quad \sup_{\Sigma_T} \left| \frac{\eta_T}{\Psi_T} \right| \lesssim \begin{cases} e^{-\frac{(n-2)T}{4}} & \text{for } n \leq 6 \\ e^{-T} & \text{for } n > 6. \end{cases}$$

This guarantees that

$$\Psi_T + \eta_T = \Psi_T \left(1 + \frac{\eta_T}{\Psi_T} \right) > 0$$

on Σ_T for sufficiently large T .

4.3. Non-linear estimates. When $\sigma_n \neq 0$, we solve the non-linear problem $\mathcal{N}_T(\Psi_T + \eta) = 0$ using a contraction mapping argument. To do this, we apply the linear estimates above, along with the following estimate of the *quadratic error term* \mathcal{Q}_T , where

$$\begin{aligned}\mathcal{Q}_T(\eta) &:= \mathcal{N}_T(\Psi_T + \eta) - (\mathcal{N}_T(\Psi_T) + \mathcal{L}_T(\eta)) \\ &= c_n \sigma_n \left(\Psi_T^{\frac{n+2}{n-2}} + \frac{n+2}{n-2} \Psi_T^{4/(n-2)} \eta - (\Psi_T + \eta)^{\frac{n+2}{n-2}} \right).\end{aligned}$$

The arguments in this subsection follow those of [13, Section 6] closely.

Lemma 4.8 (Cf. [13, Lemma 8]). *Let $k \geq 2$. For all η_i with $|\Psi_T^{-1}\eta_i| \leq \frac{1}{4}$ we have that*

$$(4.6) \quad \|\mathcal{Q}_T(\eta_1) - \mathcal{Q}_T(\eta_2)\|_{k-2,\alpha} \lesssim \left(\max_{i=1,2} \|\Psi_T^{-1}\eta_i\|_{k-2,\alpha} \right) \|\eta_1 - \eta_2\|_{k-2,\alpha}.$$

Proof. This follows at once from the expansion

$$\begin{aligned}\mathcal{Q}_T(\eta_1) - \mathcal{Q}_T(\eta_2) &= c_n \sigma_n \left[\frac{n+2}{n-2} \Psi_T^{\frac{4}{n-2}} (\eta_1 - \eta_2) + (\Psi_T + \eta_2)^{\frac{n+2}{n-2}} - (\Psi_T + \eta_1)^{\frac{n+2}{n-2}} \right] \\ &= c_n \sigma_n \frac{n+2}{n-2} (\eta_1 - \eta_2) \int_0^1 \left[\Psi_T^{\frac{4}{n-2}} - (\Psi_T + t\eta_1 + (1-t)\eta_2)^{\frac{4}{n-2}} \right] dt \\ &= c_n \sigma_n \frac{n+2}{n-2} (\eta_1 - \eta_2) \Psi_T^{\frac{4}{n-2}} \int_0^1 \left[1 - (1 + t\Psi_T^{-1}\eta_1 + (1-t)\Psi_T^{-1}\eta_2)^{\frac{4}{n-2}} \right] dt.\end{aligned}$$

□

Proposition 4.9 (Cf. [13, Proposition 9]). *Let $k \geq 2$. Let β be a constant such that $\beta \in (\frac{n-2}{4}, \frac{n-2}{2})$ when $n \leq 6$ and $\beta \in (\frac{n-2}{4}, \frac{n+2}{4})$ when $n > 6$. Let $\mathbb{B}_\beta := \{\eta \in \dot{C}^{k,\alpha}(\Sigma_T) : \|\eta\|_{k,\alpha} \leq e^{-\beta T}\}$. For sufficiently large T , the mapping*

$$F_T : \eta \mapsto -\mathcal{G}_T(\mathcal{N}_T(\Psi_T) + \mathcal{Q}_T(\eta))$$

is a contraction mapping $F_T : \mathbb{B}_\beta \rightarrow \mathbb{B}_\beta$.

Proof. Recall that “ \lesssim ” denotes an inequality up to multiplication by a constant that is independent of T . In view of the explicit expression of Ψ_T in Section 4.1, we easily see that for $\eta \in \mathbb{B}_\beta$ we have that

$$(4.7) \quad \|\Psi_T^{-1}\eta\|_{k-2,\alpha} \lesssim e^{-\beta T} e^{\frac{(n-2)}{4}T}.$$

Since $\beta > \frac{n-2}{4}$, the right-hand side of (4.7) goes to 0 uniformly as $T \rightarrow \infty$. Therefore, for any $\eta_1, \eta_2 \in \mathbb{B}_\beta$ and sufficiently large T , using the uniform bound on \mathcal{G}_T from Proposition 4.7, we have

$$\begin{aligned}\|F_T(\eta_1) - F_T(\eta_2)\|_{k,\alpha} &= \|\mathcal{G}_T(\mathcal{Q}_T(\eta_2) - \mathcal{Q}_T(\eta_1))\|_{k,\alpha} \\ &\lesssim \|\mathcal{Q}_T(\eta_2) - \mathcal{Q}_T(\eta_1)\|_{k-2,\alpha}.\end{aligned}$$

By (4.6) and (4.7), F_T is a contraction mapping on \mathbb{B}_β for T large. To see that F_T maps \mathbb{B}_β into itself, we note that by Proposition 4.6 and the upper bound for β we have that $\|\mathcal{N}_T(\Psi_T)\|_{k-2,\alpha} = o(e^{-\beta T})$, while (4.6) and (4.7) imply that $\|\mathcal{Q}_T(\eta)\|_{k-2,\alpha} = o(e^{-\beta T})$. Using once more the T -independent bound for the norm of \mathcal{G}_T from Proposition 4.7, we conclude that indeed $F_T(\eta) \in \mathbb{B}_\beta$ for $\eta \in \mathbb{B}_\beta$. □

Choose β as in Proposition 4.9. For sufficiently large T , F_T has a unique fixed point $\eta_T \in \mathbb{B}_\beta$. If we let $\hat{\Psi}_T := \Psi_T + \eta_T$, then $\mathcal{N}_T(\hat{\Psi}_T) = 0$. Since $\eta_T \in \mathbb{B}_\beta$, by (4.7), we have that $\hat{\Psi}_T > 0$ for large T . Thus we have solved the constant scalar curvature equation $R(\hat{\Psi}_T^{4/(n-2)}\gamma_T) = \sigma_n$. Moreover, by elliptic regularity, $\hat{\Psi}_T \in C^{k,\alpha}(\Sigma_T)$. Let $\hat{\gamma}_T = \hat{\Psi}_T^{4/(n-2)}\gamma_T$.

We remark that if the metrics g_i are smooth to start with, then we can bootstrap to conclude that $\hat{\gamma}_T$ is also smooth.

4.4. Volume estimate. We now derive estimates on the volume of $(\Sigma_T, \hat{\gamma}_T)$. The following proposition will complete the proof of Theorem 4.1.

Proposition 4.10. *The volume of $(\Sigma_T, \hat{\gamma}_T)$ approaches $\text{vol}(\Sigma_1, \gamma_1) + \text{vol}(\Sigma_2, \gamma_2)$ as $T \rightarrow \infty$.*

Proof. We note that $\Sigma_T \setminus ([-\frac{T}{4}, \frac{T}{4}] \times \mathbb{S}^{n-1})$ corresponds to $[\Sigma_1 \setminus B_{\gamma_1}(p_1, e^{-\frac{T}{4}}R)] \cup [\Sigma_2 \setminus B_{\gamma_2}(p_2, e^{-\frac{T}{4}}R)]$. On the respective components, we have $\Psi_i^{4/(n-2)}\gamma_T = \gamma_i$. Furthermore, $\frac{\Psi_T}{\Psi_i} \rightarrow 1$ uniformly as $T \rightarrow \infty$ on each of the respective components, and by (4.7), $\frac{\Psi_T}{\Psi_T} \rightarrow 1$ uniformly as well. Thus the volume of $(\Sigma_T \setminus ([-\frac{T}{4}, \frac{T}{4}] \times \mathbb{S}^{n-1}), \hat{\gamma}_T)$ tends to $\text{vol}(\Sigma_1, \gamma_1) + \text{vol}(\Sigma_2, \gamma_2)$.

We show now that $\text{vol}([-\frac{T}{4}, \frac{T}{4}] \times \mathbb{S}^{n-1}, \hat{\gamma}_T)$ tends to zero as $T \rightarrow \infty$. Since $\text{vol}([-1, 1] \times \mathbb{S}^{n-1}, \gamma_T)$ is uniformly bounded, by the estimate of Ψ_T , we see that $\text{vol}([-1, 1] \times \mathbb{S}^{n-1}, \Psi_T^{4/(n-2)}\gamma_T)$ goes to zero. By (4.7), we have $\left| \frac{\Psi_T}{\Psi_T} \right| \leq 2$ on Σ_T for all T sufficiently large. Thus $\text{vol}([-1, 1] \times \mathbb{S}^{n-1}, \hat{\gamma}_T)$ also goes to zero.

We next consider the right half $[1, \frac{T}{4}] \times \mathbb{S}^{n-1}$. Now, $\text{vol}([1, \frac{T}{4}] \times \mathbb{S}^{n-1}, \Psi_2^{4/(n-2)}\gamma_T)$ is less than $\text{vol}(B_{\gamma_2}(p_2, e^{-\frac{T}{4}}R), \gamma_2)$ and hence tends to 0 as $T \rightarrow \infty$. In this region, we have that $\left| \frac{\Psi_T}{\Psi_2} \right| \leq 2$ and $\left| \frac{\Psi_T}{\Psi_T} \right| \leq 2$. Hence $\text{vol}([1, \frac{T}{4}] \times \mathbb{S}^{n-1}, \Psi_2^{4/(n-2)}\hat{\gamma}_T)$ tends to 0. The left half is dealt with similarly. \square

5. LOCALIZING THE GLUING: PROOF OF THEOREM 1.6

The idea of combining the theory of local scalar curvature deformation in conjunction with a conformal gluing method in the proof here is exactly as in [4, p. 57-58].

Proof. Fix two points $p_i \in U_i$. There exists $\rho_0 > 0$ such that $B_{g_i}(p_i, \rho_0) \subset U_i$, such that for any $\rho \in (0, \rho_0)$ the operators $\Delta_{g_i} + \frac{\sigma_n}{n-1}$ have positive Dirichlet spectrum on $B_{g_i}(p_i, \rho)$, and such that $U_i \setminus \overline{B_{g_i}(p_i, \rho/2)}$ is not V -static. (The last assertion follows from an argument as in the proof of Proposition 3.1.) Fix $\rho \in (0, \rho_0)$.

Applying Theorem 4.1 with $\Sigma_i := \overline{B_{g_i}(p_i, \rho)}$, $\gamma_i := g_i$, we get a family of metrics $\{\hat{g}_T := \gamma_T\}$, with $R(\hat{g}_T) = \sigma_n$ on $\Sigma_1 \# \Sigma_2 \supset \Sigma_i \setminus B_{g_i}(p_i, \rho/2)$, such that \hat{g}_T converges to g_i in $C^{k,\alpha}(\Sigma_i \setminus B_{g_i}(p_i, \rho/2))$ and $\text{vol}(\Sigma_1 \# \Sigma_2, \hat{g}_T) \rightarrow \text{vol}(\Sigma_1, g_1) + \text{vol}(\Sigma_2, g_2)$.

We patch back in the original metric g_i , transitioning from g_i near $\partial\Sigma_i$ to \hat{g}_T near $\partial B_{g_i}(p_i, \rho/2)$ in the usual way: let $0 \leq \chi_i \leq 1$ be a fixed smooth function on M_i such that $\chi_i = 1$ in a neighborhood of $\partial\Sigma_i$ and $\chi_i = 0$ in a neighborhood of $\partial B_{g_i}(p_i, \rho/2)$. Define $\tilde{g}_T = \chi_i g_i + (1 - \chi_i) \hat{g}_T$ on $\Sigma_i \setminus B_{g_i}(p_i, \rho/2)$. Then \tilde{g}_T converges to g_i in $C^{k,\alpha}(\overline{U_i} \setminus B_{g_i}(p_i, \rho/2))$, $\text{vol}(\overline{U_1} \# \overline{U_2}, \tilde{g}_T) \rightarrow \text{vol}(\overline{U_1}, g_1) + \text{vol}(\overline{U_2}, g_2)$, and $R(\tilde{g}_T) \rightarrow \sigma_n$, with $R(\tilde{g}_T) = \sigma_n$ in a neighborhood of ∂U_i and $\partial B_{g_i}(p_i, \rho/2)$. Since g_i is not V -static on $U_i \setminus \overline{B_{g_i}(p_i, \rho/2)}$, Theorem 1.2 can now be applied on $U_i \setminus \overline{B_{g_i}(p_i, \rho/2)}$

to deform \tilde{g}_T (for sufficiently large T) to a metric \tilde{g} such that $(\tilde{g}_T - \tilde{g})$ has compact support in $U_i \setminus \overline{B_{g_i}(p_i, \rho/2)}$, $R(\tilde{g}) = \sigma_n$, and

$$\begin{aligned} \text{vol}(\overline{U}_1, g_1) + \text{vol}(\overline{U}_2, g_2) &= \text{vol}(\overline{U}_1 \setminus B_{g_1}(p_1, \rho/2), \tilde{g}) + \text{vol}(\overline{U}_2 \setminus B_{g_2}(p_2, \rho/2), \tilde{g}) \\ &\quad + \text{vol}(\overline{U}_1 \# \overline{U}_2 \setminus ((\overline{U}_1 \setminus B_{g_1}(p_1, \rho/2)) \sqcup (\overline{U}_2 \setminus B_{g_2}(p_2, \rho/2))), \hat{g}_T). \end{aligned}$$

The metric g on $M_1 \# M_2$ given by $g = g_i$ on $M_i \setminus U_i$, $g = \tilde{g}$ on $\overline{U}_i \setminus \overline{B_{g_i}(p_i, \rho/2)}$, and $g = \hat{g}_T$ on $(\overline{U}_1 \# \overline{U}_2) \setminus ((\overline{U}_1 \setminus B_{g_1}(p_1, \rho/2)) \sqcup (\overline{U}_2 \setminus B_{g_2}(p_2, \rho/2)))$ has all the properties asserted in Theorem 1.6. \square

6. COUNTEREXAMPLES TO MIN-OO'S CONJECTURE WITH NON-TRIVIAL TOPOLOGY AND ARBITRARILY LARGE VOLUME

In [23], Min-Oo conjectured that if $(\overline{\Omega}, g)$ is a compact Riemannian manifold with boundary such that g has scalar curvature at least $n(n-1)$, such that $\partial\Omega$ is isometric to the standard round sphere \mathbb{S}^{n-1} , and such that $\partial\Omega$ is totally geodesic in $(\overline{\Omega}, g)$, then $(\overline{\Omega}, g)$ is isometric to the standard round hemisphere \mathbb{S}_+^n . Various affirmative partial results under stronger hypotheses have been achieved in this direction. We refer the reader to [3] for a comprehensive account of these contributions. Recently, Brendle, Marques, and Neves constructed a counterexample to Min-Oo's conjecture:

Theorem 6.1 ([3, Theorem 7]). *Given any integer $n \geq 3$, there exists a smooth metric g on the hemisphere \mathbb{S}_+^n with the following properties:*

- (1) *The scalar curvature of g is at least $n(n-1)$ at every point on \mathbb{S}_+^n .*
- (2) *The scalar curvature of g is strictly greater than $n(n-1)$ at some point on \mathbb{S}_+^n .*
- (3) *The metric g agrees with the standard round metric on \mathbb{S}_+^n in a neighborhood of the equator $\partial\mathbb{S}_+^n$.*

We can construct more complicated counterexamples to Min-Oo's conjecture from counterexamples such as these by combining Theorem 1.2 and Theorem 4.1 (see also Remark 6.4):

Proposition 6.2. *Let g be a metric on \mathbb{S}_+^n that is a counterexample to Min-Oo's conjecture. Suppose g agrees with the standard round metric \bar{g} in a neighborhood of the equator $\partial\mathbb{S}_+^n$. Given any constant $V_0 > 0$, there exists a counterexample $(\mathbb{S}_+^n, \hat{g})$ to Min-Oo's conjecture such that $\text{vol}(\mathbb{S}_+^n, \hat{g}) \geq V_0$.*

Proof. By analyticity, cf. the comments following (2.2), the metric g cannot be V -static on \mathbb{S}_+^n .

Let $B_g(p, \rho) \subset \mathbb{S}_+^n$ be a geodesic ball such that $R(g) = n(n-1)$ on $B_g(p, \rho)$, $\Delta_g + n$ has positive Dirichlet spectrum on $B_g(p, \rho)$, and g is not V -static on $\mathbb{S}_+^n \setminus \overline{B_g(p, \rho/2)}$. We can proceed exactly as in the proof of Theorem 1.6 and glue (\mathbb{S}_+^n, g) to a copy of itself, first applying Theorem 4.1 in case Σ_1 and Σ_2 are taken to be $\overline{B_g(p, \rho)}$ from each copy, and then applying Theorem 1.2 to each copy of the non-static region $\mathbb{S}_+^n \setminus \overline{B_g(p, \rho/2)}$. The resulting metric \hat{g} on $\mathbb{S}_+^n \# \mathbb{S}_+^n$ agrees with g to infinite order at $\partial\mathbb{S}_+^n$ in both copies of \mathbb{S}_+^n . Moreover, $R(\hat{g}) = n(n-1)$ in the neck region while $R(\hat{g}) = R(g)$ on $\mathbb{S}_+^n \setminus \overline{B_g(p, \rho/2)}$ in both copies of \mathbb{S}_+^n , and $\text{vol}(\mathbb{S}_+^n \# \mathbb{S}_+^n, \hat{g}) = 2\text{vol}(\mathbb{S}_+^n, g)$. Because g coincides with the standard round metric \bar{g} near $\partial\mathbb{S}_+^n$ and \hat{g} extends smoothly to \bar{g} across $\partial\mathbb{S}_+^n$, we can then add a standard round hemisphere to one of the two copies of \mathbb{S}_+^n . Clearly, this process can be iterated, increasing the volume at every stage by a fixed amount. \square

In conjunction with Theorem 6.1, it follows that there are counterexamples to Min-Oo's conjecture of arbitrarily large volume. In contrast, it is shown in [22] that a metric g on \mathbb{S}_+^n that satisfies conclusions (1) and (3) of Theorem 6.1 and which is also C^2 -close to the standard round metric \bar{g} on \mathbb{S}_+^n has volume less than $\text{vol}(\mathbb{S}_+^n, \bar{g})$.

Remark 6.3. In the proof of Proposition 6.2, we can arrange that \hat{g} agrees with the round metric \bar{g} near $\partial\mathbb{S}_+^n$. Indeed, when applying the proof of Theorem 1.2, we can first find a collar neighborhood $N \subset \mathbb{S}_+^n$ of $\partial\mathbb{S}_+^n$, so that $B_g(p, \rho) \subset U := \mathbb{S}_+^n \setminus \overline{N}$, and $U \setminus \overline{B_g(p, \rho/2)}$ is not V -static. We also note that the proof of Proposition 6.2 can be applied to any counterexample of the Min-Oo conjecture which is not V -static and contains a domain with constant scalar curvature. We can connect to such a space one of the examples satisfying conditions (1) and (3) of Theorem 6.1. After applying Theorem 1.6, we can cap off this end with a round sphere as above.

Remark 6.4. The large-volume counterexamples to Min-Oo's conjecture can alternatively be obtained from the Brendle-Marques-Neves counterexample using the Gromov-Lawson connect-sum construction for positive scalar curvature [12]. The construction in [12] is local near the gluing points. One would connect two copies of the example of Brendle-Marques-Neves at points where $R(g) > n(n-1)$, applying the technique of Gromov-Lawson carefully so as to maintain the lower bound on the scalar curvature.

Remark 6.5. In the proof of Proposition 6.2, one can start with two disjoint small balls $B_g(p_1, \rho_1)$ and $B_g(p_2, \rho_2)$ in (\mathbb{S}_+^n, g) such that $R(g) = n(n-1)$ on $B_g(p_i, \rho_i)$, $\Delta_g + n$ has positive Dirichlet spectrum on $B_g(p_i, \rho_i)$, $i = 1, 2$, and g is not V -static on $U = \mathbb{S}_+^n \setminus (\overline{B_g(p_1, \frac{\rho_1}{2})} \cup \overline{B_g(p_2, \frac{\rho_2}{2})})$. By forming the connected sum of $B_g(p_1, \rho_1)$ and $B_g(p_2, \rho_2)$ (adding a handle) and deforming the metric on U , one obtains a counterexample to Min-Oo's conjecture with non-trivial fundamental group. One can also obtain such an example by connecting a counterexample to Min-Oo's conjecture to a non- V -static metric on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ or \mathbb{S}^n/Γ (where Γ is a finite subgroup of $SO(n+1)$) which has scalar curvature at least $n(n-1)$, and in some region has constant scalar curvature $n(n-1)$. The existence of such metrics on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ or \mathbb{S}^n/Γ follows from results of Kazdan-Warner. In fact, it is shown in [16, 17] that on every closed manifold that admits a smooth metric of positive scalar curvature, every smooth function is the scalar curvature of some smooth metric. Applying the proof of Proposition 6.2 to such an example, one obtains more complicated counterexamples with non-trivial topology and arbitrarily large volume.

APPENDIX A. SCHAUDER THEORY

Here we discuss interior Schauder estimates in weighted spaces, following Chruściel and Delay [4, Appendix B], for the particular example of the operator $u \mapsto P(u) := \rho^{-1} L_g \rho L_g^* u$. Note that, in local coordinates, $P(u)$ has the form

$$(n-1)\Delta_g^2 u + \sum_{|\beta| \leq 3} b_\beta D^\beta u.$$

Recall that the weight ρ is a smooth function that behaves like $e^{-1/d}$ near $\partial\Omega$. It is easy to check that $\|b_\beta\|_{C_{\phi, \phi^{4-|\beta|}}^{0, \alpha}} < \infty$. By appropriate scaling, one can obtain interior Schauder estimates on small balls near the boundary of Ω from interior Schauder estimates on balls of a fixed size for an operator whose coefficients are

well controlled. The weighted Hölder norms defined in Section 3.1 are designed for this purpose.

For simplicity, we assume that we are working in \mathbb{R}^n with the standard metric, and that x is close to $\partial\Omega$ so that $\phi(x) = d(x)^2$. For $z \in B(0, 1)$, let $y = x + \phi(x)z$, and for any f , let $\tilde{f}(z) = f(x + \phi(x)z) = f(y)$. Then $D_z \tilde{u}|_z = \phi(x) D_y u|_{x+\phi(x)z}$. We compute that

$$\begin{aligned} (Pu)(x + \phi(x)z) &= (n-1)\Delta_y^2 u|_{x+\phi(x)z} + \sum_{|\beta| \leq 3} b_\beta D_y^\beta u|_{x+\phi(x)z} \\ &= \phi(x)^{-4}(n-1)\Delta_z^2 \tilde{u}|_z + \sum_{|\beta| \leq 3} \phi(x)^{-|\beta|} \tilde{b}_\beta(z) D_z^\beta \tilde{u}|_z. \end{aligned}$$

We obtain that

$$\phi(x)^4 \tilde{P}u(z) = \left((n-1)\Delta_z^2 + \sum_{|\beta| \leq 3} \phi(x)^{4-|\beta|} \tilde{b}_\beta(z) D_z^\beta \right) \tilde{u} =: \tilde{P}\tilde{u}(z).$$

We see that \tilde{P} is uniformly elliptic on $B(0, 1)$ and has coefficients that are bounded in $C^{0,\alpha}$ by bounds for $\|b_\beta\|_{C^{0,\alpha}_{\phi, \phi^{4-|\beta|}}}$. The standard interior Schauder estimate gives

$$\begin{aligned} \|\tilde{u}\|_{C^{4,\alpha}(B(0, \frac{1}{4}))} &\leq C \left(\|\tilde{P}\tilde{u}\|_{C^{0,\alpha}(B(0, \frac{1}{2}))} + \|\tilde{u}\|_{L^2(B(0, \frac{1}{2}))} \right) \\ &\leq C \left(\phi(x)^4 \|\tilde{P}u\|_{C^{0,\alpha}(B(0, \frac{1}{2}))} + \|\tilde{u}\|_{L^2(B(0, \frac{1}{2}))} \right). \end{aligned}$$

Scaling back, we see that

$$\begin{aligned} &\sum_{j=0}^4 \phi(x)^j \|\nabla_g^j u\|_{C^{0,\alpha}(B(x, \frac{\phi(x)}{4}))} + \phi(x)^{4+\alpha} [\nabla_g^4 u]_{0,\alpha; B(x, \frac{\phi(x)}{4})} \\ &\leq C \left(\phi(x)^4 \|Pu\|_{C^0(B(x, \frac{\phi(x)}{2}))} + \phi(x)^{4+\alpha} [Pu]_{0,\alpha; B(x, \frac{\phi(x)}{2})} + \phi(x)^{-\frac{n}{2}} \|u\|_{L^2(B(x, \frac{\phi(x)}{2}))} \right). \end{aligned}$$

We can multiply this inequality by $\varphi(x)$ where $\varphi = \phi^r \rho^s$ to obtain the following weighted estimate on Ω :

$$(A.1) \quad \|u\|_{C^{4,\alpha}_{\phi, \varphi}} \leq C(\|Pu\|_{C^{0,\alpha}_{\phi, \phi^4 \varphi}} + \|u\|_{L^2_{\phi^{-n} \varphi^2}}).$$

This estimate is similar to that in [4, Appendix B]. Note that we impose slightly different conditions on the lower order coefficients here, and that we use a different convention for the weighted L^2 norms. As for higher regularity, we obtain similarly that

$$(A.2) \quad \|u\|_{C^{k+4,\alpha}_{\phi, \varphi}} \leq C(\|Pu\|_{C^{k,\alpha}_{\phi, \phi^4 \varphi}} + \|u\|_{L^2_{\phi^{-n} \varphi^2}})$$

where the constant C depends on the domain, the weight, and bounds for $\|b_\beta\|_{C^{k,\alpha}_{\phi, \phi^{4-|\beta|}}}$.

APPENDIX B. PROOF OF PROPOSITION 4.6

Here we sketch the proof of Proposition 4.6, which is similar to that of [13, Proposition 6] and [14, Proposition 3.6].

Recall that $[-\frac{T}{2}, \frac{T}{2}] \times \mathbb{S}^{n-1} \cong C_T \subset \Sigma_T$. In the proof below, Hölder norms on C_T or $Q := [-1, 1] \times \mathbb{S}^{n-1} \subset C_T$ are indicated with an additional subscript.

Proof. We recall that on $\Sigma_T \setminus C_T$, $\mathcal{N}_T(\Psi_T) = 0$, and that on $[-\frac{T}{2} + 1, \frac{T}{2} - 1] \times \mathbb{S}^{n-1}$, $\Psi_T(s, \theta) = 2R^{\frac{n-2}{2}} e^{-\frac{(n-2)T}{4}} \cosh\left(\frac{(n-2)s}{2}\right)$. Let $\hat{\gamma} = ds^2 + g_{\mathbb{S}^{n-1}}$ be the standard cylindrical metric. Then Ψ_T solves the equation $(-\Delta_{\hat{\gamma}} + c_n R(\hat{\gamma}))(\Psi_T) = 0$ on $[-\frac{T}{2} + 1, \frac{T}{2} - 1] \times \mathbb{S}^{n-1}$. Therefore, by (4.1) and Lemma 4.3, we have

$$(B.1) \quad \begin{aligned} \|\Delta_{\gamma_T} f - \Delta_{\hat{\gamma}} f\|_{k-2, \alpha, C_T} &\lesssim e^{-T} \cosh 2s \|f\|_{k, \alpha; C_T} \\ \|R(\gamma_T) - R(\hat{\gamma})\|_{k-2, \alpha, C_T} &\lesssim e^{-T} \cosh 2s. \end{aligned}$$

On $Q \cong [-1, 1] \times \mathbb{S}^{n-1}$, (B.1) implies $\|\mathcal{N}_T(\Psi_T)\|_{k-2, \alpha, Q} \lesssim e^{-\frac{(n+2)T}{4}}$. This completes the estimate on Q .

We now consider $C_T \setminus Q \cong ([-\frac{T}{2}, -1] \times \mathbb{S}^{n-1}) \cup ([1, \frac{T}{2}] \times \mathbb{S}^{n-1})$. The estimates on the two components are similar. We will do one of them.

Recall that on $[1, \frac{T}{2}] \times \mathbb{S}^{n-1}$ we have that $\gamma_T = \Psi_2^{-4/(n-2)} \gamma_2$ and $\Psi_T(s, \theta) = \chi_{1,T}(s) \psi(R e^{-s-\frac{T}{2}}) + \chi_{2,T}(s) \psi(R e^{s-\frac{T}{2}})$. Moreover, $\mathcal{N}_T(\Psi_2) = 0$ in this region, so that

$$\mathcal{N}_T(\Psi_T) = (-\Delta_{\gamma_T} + c_n R(\gamma_T))(\chi_{1,T} \Psi_1) - c_n \sigma_n \Psi_T^{\frac{n+2}{n-2}} + c_n \sigma_n \Psi_2^{\frac{n+2}{n-2}}.$$

We write the last two terms using $\Psi_T^{\frac{n+2}{n-2}} - \Psi_2^{\frac{n+2}{n-2}} = \Psi_2^{\frac{n+2}{n-2}} \left(\left(1 + \chi_{1,T} \frac{\Psi_1}{\Psi_2}\right)^{\frac{n+2}{n-2}} - 1 \right)$.

Since also $\frac{\Psi_1}{\Psi_2} = e^{-s(n-2)}$ and $\Psi_2^{\frac{n+2}{n-2}} = (R e^{s-\frac{T}{2}})^{\frac{n+2}{2}}$ in this region, we obtain that

$$(B.2) \quad \left| \Psi_T^{\frac{n+2}{n-2}} - \Psi_2^{\frac{n+2}{n-2}} \right| \lesssim \Psi_2^{\frac{n+2}{n-2}} \cdot \frac{\Psi_1}{\Psi_2} \lesssim e^{-s \frac{n-6}{2}} e^{-\frac{(n+2)T}{4}}.$$

On $[1, \frac{T}{2} - 1] \times \mathbb{S}^{n-1}$, (B.2) shows

$$\left| \Psi_T^{\frac{n+2}{n-2}} - \Psi_2^{\frac{n+2}{n-2}} \right| \lesssim \begin{cases} e^{-\frac{(n+2)T}{4}} & \text{for } n > 6 \\ e^{-\frac{(n-2)T}{2}} & \text{for } n \leq 6 \end{cases}$$

while on $[\frac{T}{2} - 1, \frac{T}{2}] \times \mathbb{S}^{n-1}$, (B.2) gives that

$$\left| \Psi_T^{\frac{n+2}{n-2}} - \Psi_2^{\frac{n+2}{n-2}} \right| \lesssim e^{-\frac{(n-2)T}{2}}.$$

The required Hölder bounds of $\|\Psi_T^{\frac{n+2}{n-2}} - \Psi_2^{\frac{n+2}{n-2}}\|_{k-2, \alpha, C_T}$ follow analogously.

It remains to estimate $\|(-\Delta_{\gamma_T} + c_n R(\gamma_T))(\chi_{1,T} \Psi_1)\|_{k-2, \alpha, C_T}$. We first estimate on $[1, \frac{T}{2} - 1] \times \mathbb{S}^{n-1}$, where $\chi_{1,T} = 1$. Using this along with (B.1) and the fact that Ψ_1 is in the kernel of the conformal Laplacian on the cylinder, we obtain

$$\begin{aligned} &\left| (-\Delta_{\gamma_T} + c_n R(\gamma_T))(\chi_{1,T} \Psi_1) \right| \\ &= \left| (-\Delta_{\gamma_T} + c_n R(\gamma_T))(\Psi_1) - (-\Delta_{\hat{\gamma}} + c_n R(\hat{\gamma}))(\Psi_1) \right| \\ &\lesssim e^{-T} \cosh(2s) \|\Psi_1\|_{C^2(C_T)} \\ &\lesssim e^{s(2-\frac{n-2}{2})} e^{-T} e^{-\frac{(n-2)T}{4}} \\ &\lesssim \begin{cases} e^{-T} e^{-\frac{(n-2)T}{4}} = e^{-\frac{(n+2)T}{4}} & \text{for } n > 6 \\ e^{(1-\frac{n-2}{4})T} e^{-T} e^{-\frac{(n-2)T}{4}} = e^{-\frac{(n-2)T}{2}} & \text{for } n \leq 6. \end{cases} \end{aligned}$$

For $s \in [\frac{T}{2} - 1, \frac{T}{2}]$, using that $\Psi_1 = (Re^{-s})^{\frac{n-2}{2}} e^{-\frac{(n-2)T}{4}}$, we have

$$\begin{aligned} & \left| (-\Delta_{\gamma_T} + c_n R(\gamma_T))(\chi_{1,T} \Psi_1) \right| \\ & \lesssim e^{-T} \cosh(2s) \|\Psi_1\|_{C^2(C_T)} + \left| (-\Delta_{\dot{\gamma}} + c_n R(\dot{\gamma}))(\chi_{1,T} \Psi_1) \right| \lesssim e^{-\frac{(n-2)T}{2}}. \end{aligned}$$

This proves the desired C^0 bound in (4.3) and (4.4). The estimate of the derivatives and the Hölder bound follow from similar reasoning. \square

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